1 OVERVIEW

1.1 Introduction

1.1.1 A Penny for Thought

Imagine you own a marriage agency. Recently, there are four bachelors named Andy, Benny, Canny and Danny, who are all contemplating marriage. Just as the great philosopher Socrates once said, “Marry, if you get a good wife, you’ll be happy. If not, you’ll become a each man is worried about finding the right mate. As you have a well-known reputation as a matchmaker, these four men approach you. Therefore, you introduce them to Anna, Betty, Candy, and Donna. After the meeting, each person ranks every member of the opposite sex, and submits the rankings to you.

Suppose the rank-ordered preference lists are as follows:

<table>
<thead>
<tr>
<th>Men's preference lists</th>
<th>Women's preference lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andy</td>
<td>Anna  Betty  Candy  Donna</td>
</tr>
<tr>
<td>Benny</td>
<td>Betty  Donna  Anna  Candy</td>
</tr>
<tr>
<td>Canny</td>
<td>Donna  Anna  Candy  Betty</td>
</tr>
<tr>
<td>Danny</td>
<td>Betty  Anna  Candy  Donna</td>
</tr>
</tbody>
</table>

For example, Andy likes Anna most and Donna least while Anna likes Benny most and Andy least. Given the above preference lists, your job is to find a match for each party. As your reputation depends on the number of successful marriage performed, what should you do?¹

It is much easier to think about what you should not do. Suppose you come up with the following recommendation: Match Andy with Anna, Benny with Betty, Canny with Donna and Danny with Candy. Notice that Danny prefers Betty to Candy, his recommended partner, while Betty prefers Danny to Benny, her recommended partner. It is not hard to guess that your
proposed marriage would “break up” as Danny and Betty would prefer each other than to stick to your recommendation.

The above proposed marriage is said to be “unstable” as there is a pair of man and woman who can both benefit by marrying each other, rather than keeping their original matches intact. Such situation is certainly undesirable for you as a matchmaker. Thus, we can conclude that the least you should do is to search for a stable marriage where no pair of man and woman will find it beneficial to “divorce” their respective partners and marry each other.

1.1.2 Background

The scenario given earlier is a type of the stable marriage problem\(^2\) that forms the basis of our study. The general theme of our study is the two-sided matching markets, which include other stable matching problems as well. The two-sided matching problem has held a fascination for many computer scientists, mathematicians and economists, among others, ever since its introduction in the founding paper of Gale and Shapley (1962). Research conducted during the past thirty-six years has helped us understand and appreciate its connections to a variety of problems arising in combinatorics, operation research and economics.

1.1.2.1 Two-sided, matchings and stability

Two-sided:

As illustrated in the given scenario, the term “two-sided” in the “two-sided matching markets” refers to the fact that agents in such markets belong to one of two disjoint sets -- e.g. 1.)

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\(^1\) This problem originates from V.J Sethuraman and C.P.Teo, Linear programming brings marital bliss. To appear in Singapore Mathematical Medly, 1998.

\(^2\) It is often cited in the literature that the stable marriage problem is perhaps the simplest kind of the two-sided matching markets.
man or woman in the stable marriage problem; and 2.) hospitals or residents in the hospitals/residents problem. This differs from some financial markets such as the stock market whereby an agent can take the role of a buyer or a seller depending on the prevailing market price. Hence, financial markets are not considered to be two-sided in our context.

**Matchings:**

The term “matching” generally implies that the two disjoint sets are to be matched under certain criterion. The criterion in question is that of stability, and it depends entirely on the preferences expressed by the agents.

**Framework of stable matching:**

Generally, a matching is said to be *unstable* if there are two agents who are not matched to each other but prefer to be together. Hence, a *stable* matching is one that *is not unstable*. The importance of stability in two sided matching markets is best highlighted by a system where the acceptance of the proposed matching is voluntary. In such a system, an unstable matching will not remain intact as the unmatched pair(s) will opt to be together, just like what Danny and Betty would do in the given scenario.

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3 As illustrated in the given scenario, a matching in a stable marriage problem is simply a one-one mapping between the men and the women. However, in the hospitals/residents problem, a matching is a many to one mapping between the residents and the hospitals with each of the latter being assigned to a group of residents whose size is within the quotas of residents it can admit. A common terminology for the hospitals/residents problem is the college admission problem. In the college admission problem, the two discrete sets are the students and the colleges. Like the hospitals in the hospitals/residents problem, the colleges have some “fixed” number of places to fill. A matching in the college admission problem is an assignment of students to colleges so that no college exceeds its number of available places. For consistency, we would use the terminology of the hospitals/residents problem in our study.
1.1.3 Pioneers in the Research on Two-Sided Matching Markets- Gale and Shapley

In their paper entitled “College Admissions Problem and the Stability of Marriage”, Gale and Shapley (1962) showed that we could always find a stable matching for any instance of the stable marriage problem. The authors continued to show that we can always find a man-optimal stable matching that simultaneously gives all men their best match they can possibly have in any stable matching, and another woman-optimal stable matching, which gives all women their best match they can possibly have in any stable matching. They proved these results using a simple algorithm that constructed the man-optimal stable matching. This algorithm is commonly known as the man-propose algorithm because it can be expressed as a sequence of “proposals” from the men to the women. This proposed algorithm was later generalised to deal with the hospitals/residents problem.

1.2 Justifications

1.2.1 Beyond Gale and Shapley

As far as the stable marriage problem is concerned, Gale and Shapley’s paper in 1962 might have been expected to mark an end to the whole matter. Surprisingly, it merely sets the stage for further extensive study in the two-sided matching markets, and in particular, the stable marriage problem.

Many other variants of the stable marriage problem have been introduced and studied. These problems have proven to be of perpetual engrossment to people working in a wide range of disciplines that includes the computer science, economics, game theory, management science and operations research.

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4 The man-propose algorithm can be easily converted to the woman-propose algorithm that can be used to compute the woman-optimal stable matching. This is accomplished by reversing the sex roles in the man-propose algorithm.
1.2.1.1 Developments in the study of stable marriage problem

Today, there are many different approaches to the study of the stable marriage problem. One approach, developed principally by the computer scientists, has been to explore the structural aspects of the stable marriage problem. The research has enhanced our understanding of the stable marriage problem and has resulted in the discovery of several fascinating properties of the stable matching solutions.

A vastly different line of investigation, which has been developed principally by mathematical economists and game theorists, focuses on the incentives each agent has in falsifying his/her preferences. This approach seeks to understand and quantify the potential gains of a deceitful agent. There are numerous ways to model this possibility, one of which is to assume that the deceitful agent has complete knowledge about the true preferences of the other agents. Under this assumption, several intriguing results that have great importance for economics and game theory have been developed.

It was found by Roth (1982) that when the man-propose algorithm is used, no man would be better off by falsifying his preferences.\(^5\) Falsifying preferences will only help each man to retain his original partner whom he already gets when he reveals his true preferences. We can conclude that each man will never have the incentive to falsify his preferences when the man-propose algorithm is used; it is a dominant strategy for him to state his true preferences.

However, the same conclusion cannot be applied to every woman. Gale and Sotomayor (1985b) showed that when the man-propose algorithm is used, some women would have the incentive to lie about their preferences. In fact, it is shown that every woman will always have

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\(^5\) Roth’s result was stated in terms of a stable matching procedure that always yields the optimal stable outcome for all the men. From the point of view of incentives, any two procedures are equivalent. Since the man-propose algorithm also yields the man-optimal stable matching, we shall just express Roth’s result in terms of the man-propose algorithm.
the incentive to cheat as long as she does not get her best stable partner (i.e. her woman-optimal partner) when the man-propose algorithm is used. This is because she could always get her woman-optimal partner by falsifying her preferences. According to Gale and Sotomayor (1985b), all she needs to do is to state a preference list that ranks the men in the same orders as in her true preference list, but ranks as unacceptable all men who are ranked below her woman-optimal stable partner.\footnote{When a woman declares a man as unacceptable, it means that she would rather remain single than to be matched to the man. If that is the case, her preference list would not contain the man. In other words, her preference list would only include those men that she would prefer to marry.}

Indeed, this is an “optimal” lying strategy for the women to adopt. When the man-propose algorithm is used, every woman will get her worst stable partner. By just following the strategy, a deceitful woman can always escape from her fate of getting her worst stable partner. What’s more, she can even force the man-propose algorithm to give her her woman-optimal stable partner.

However, prior to our research, we know of no analogous general results when the women are required to submit a complete preference list, i.e. the women cannot opt to remain single in the matching game. Interestingly, this restriction, which is the original model studied by Gale and Shapley (i.e. a model without rejection), changes the complexity of the problem as it is not known how a woman could lie by permuting her preferences under the man-propose algorithm. In fact, it is also unclear about what can be achieved by lies that merely permute the original order of the men on a woman’s list. We note that this version of the matching problem (i.e. opting out is not permitted) is present in many college-student matching markets (although it is a many-to-one matching market) as schooling is compulsory in certain countries (e.g. Singapore).
Josh Benaloh [cf. Gusfield and Irving (1989)] found the following example which shows that the women can lie by permuting their preference lists to force the man-propose algorithm to give them their woman-optimal partners.

Example 1.1: Lying by permuting can be beneficial

A) When the true preferences are stated:

<table>
<thead>
<tr>
<th></th>
<th>Women's preferences</th>
<th>Men's preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 1 3</td>
<td>1 1 2 3</td>
</tr>
<tr>
<td>2</td>
<td>1 2 3 1</td>
<td>2 2 1 3</td>
</tr>
<tr>
<td>3</td>
<td>1 2 3 1</td>
<td>3 1 2 3</td>
</tr>
</tbody>
</table>

When the man-propose algorithm is used, the man-optimal stable matching is \{(man 1, woman 1), (man 2, woman 2), (man 3, woman 3)\}, with man 1 matched to woman 1 (her second choice).

B) When woman 1 changes her preference list to

<table>
<thead>
<tr>
<th></th>
<th>Women's preferences</th>
<th>Men's preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 2 3 1</td>
<td>1 1 2 3</td>
</tr>
<tr>
<td>2</td>
<td>1 2 3 1</td>
<td>2 2 1 3</td>
</tr>
<tr>
<td>3</td>
<td>1 2 3 1</td>
<td>3 1 2 3</td>
</tr>
</tbody>
</table>

The man-optimal stable matching is now \{(woman 1, man 2), (woman 2, man 1), (woman 3, man 3)\}, with woman 1 now matched with man 2 (her true first choice).

Based on the above observations, it is reasonable to ask whether there is a cheating strategy, analogous to that proposed by Gale and Sotomayor (1985b), which a deceitful agent can follow when it is not a dominant strategy for him (her) to state his (her) true preferences and given that he (she) can only lie by permuting his (her) preferences. Furthermore, can a woman always force the man-propose algorithm to return her woman-optimal stable partner in this model? These are some of the strategic issues in the Gale-Shapley model, which this research seeks to address.
1.2.1.2 Practical applications of Gale and Shapley’s work

Surprisingly, the ideas suggested by Gale and Shapley (1962) had actually been applied in 1952, which predates their work by 10 years. The National Resident Matching Program, NRMP (located in Evanston, Illinois), has the task to assign graduating medical students (residents) for internship in the country’s hospitals. The method used by NRMP is a polygamous version of the Gale and Shapley algorithm, which produces the hospital-optimal stable matching.

Other complex forms of two-sided matching markets have also been identified and studied in depth. Famous examples include, other than the U.S internship market [Roth (1984), (1990)], the U.K internship market [Roth (1990)] and the U.S Naval Academy market [New York Times, January 30, 1986, page 8].

From the basic stable marriage model, many new models have been derived that are purported to explain the actual operations in these real markets. For example, it is originally assumed that the hospitals/residents problem can always be reduced to a stable marriage problem. The inherent implication is that the results for the marriage problem could be generalized to the hospitals/residents problem and any other similar many-to-one matching markets. Hence, the stable marriage model should be effective in explaining the actual operations in the U.S internship market. However, Roth [(1982), (1986)] found that some of the results, established for the marriage problem, do not generalized to the hospitals/residents problem. Consequently, this leads to a reformulation of the hospitals/residents problem that is purported to explain and predict correctly the actual operations in the internship market.
1.2. 2 A Locally Found Two-sided Matching Market

In the existing literature, all the two-sided matching markets examined are found abroad. It would be interesting to examine a two-sided matching market found in Singapore. One excellent candidate is the primary six pupils/secondary schools market, which concerns the assignment of pupils to secondary schools upon completion of their Primary School Leaving Examinations (PSLE).

Unlike those two-sided matching markets that we have come across in the existing literature, this is a market where opting out is not permitted and it is compulsory for every pupil to be assigned to a school upon the completion of his/her PSLE.

Undeniably, the Ministry of Education (MOE) has certainly done an excellent job in ensuring that the primary six pupils are assigned to the secondary schools in the fairest possible manner. Interestingly, the posting method used by the MOE is actually a form of student-propose algorithm. According to Roth (1982), each pupil will never have the incentive to lie about his/her true preferences. However, in reality, this may not be the case where some pupils may actually be induced to tell lies. Why is it so? This is one issue that this research seeks to elucidate.

1.3 Objectives

The preceding section outlines the motivation for our research. Basically, our research can be split into two parts:

1.3.1 Strategic Issues in Gale-Shapley Stable Marriage Game

This part of our research focuses on the theoretical aspects of the stable marriage problem where we are examining the strategic issues involved under the assumption that no agent is
allowed to declare any member of the opposite set as unacceptable (i.e. each agent must submit a complete preference list and thus he/she is only permitted to lie by permuting his/her preference).

1.3.1.1 Main objectives

a.) To investigate the incentives facing each agent (in this case, man/woman) in revealing his/her true preference lists.

b.) To come up with an optimal lying strategy for an agent, who finds it beneficial to lie about his/her preferences.

c.) To identify the similarities and differences in results relating to strategic issues in the stable marriage problem when i.) The agents are allowed to declare any member of opposite sex to be unacceptable, and ii.) The agents are not allowed to declare any member of opposite sex to be unacceptable.

1.3.2 Study of Two-Sided Matching Market in Singapore

This part of our research looks at the primary six pupils /secondary schools market, a two-sided matching market found locally. In this study, the current posting system adopted by the MOE is analyzed.

1.3.2.1 Main objectives

a.) To understand the strategic behavior of the pupils and schools.

b.) To identify the problems of the current posting system posed for the pupils, schools and MOE.

c.) To make recommendations to improve the current posting algorithm.
Last but not the least, the Academic Exercise serves to provide a detailed coverage on the work done so far on two-sided matching markets. The objective is to provide the readers and those who may be interested in doing a study on two-sided matching markets a better understanding on the topic itself.

### 1.4 Organization of Research

This Academic Exercise is divided into 5 chapters. In the next chapter, literature pertaining to our research will be reviewed. Chapter three marks the first part of our research. In that chapter, we examine the strategic issues in Gale-Shapley stable marriage game. The second part of our research is found in chapter four. In that chapter, we analyze the Secondary One Posting Exercise, and identify the problems it poses for the schools, pupils and the MOE. Finally, chapter five wraps up our research with a conclusion and a review of future research in two-sided matching markets.
2 LITERATURE REVIEW

2.1 Introduction

Since their introduction by Gale and Shapley in 1962, two-sided matching markets have captured the interests of people working in diverse areas. A very rich literature based on the problems soon evolved, which by now includes at least three books and more than a hundred papers.

In this chapter, a modest attempt is made to provide a comprehensive literature review that is relevant to our Academic Exercise. The purpose of this literature review is to provide the readers a better understanding of two-sided matching markets.

2.2 Organization

From sections 2.3-2.7, the review is on the stable marriage problem. This is followed by a review on the hospitals/residents problem in section 2.8. In section 2.3, we review the pioneering paper of Gale and Shapley in more details. In additional, we also explore the basic Gale and Shapley algorithm for the stable marriage problem, together with some of its implications. In section 2.4, we provide an overview on the essential structural properties of the stable marriage problem. In section 2.5, we briefly discuss the three most common variants of the stable marriage problem, highlighting the major differences between each variant and the basic problem. In section 2.6, we give an account of the findings on the strategic behaviour of the agents involved in the stable marriage problem. Next, in section 2.7 we document some well-known applications of the stable marriage problem. Last but not the least, in section 2.8, we cover in some details the hospitals/residents problem, highlighting the major differences between the problem itself and the stable marriage problem.
2.3 Gale and Shapley’s Paper

Gale and Shapely’s original problem is a college admission problem with n students and m colleges; each of the latter has its own quota of students for admission. In the case that each college has a quota of one and there are equal number of colleges and students, the problem reduces to a stable marriage problem. Gale and Shapley formulated the classical stable marriage problem of matching equal number of men and women, each of whom has a strictly rank ordered preference list (i.e., no ties allowed) over every member of the opposite sex. By means of an algorithm, this marriage problem is solved, thereby proving the fact that a stable matching does exist.\(^7\)

2.3.1 The Basic Gale-Shapley Algorithm (equivalently, Man-propose Algorithm)

Informally, this algorithm can be expressed as a sequence of steps with the men proposing to the women at each step. The following is a brief description of how the algorithm works.

At the start of the algorithm, each man proposes to his favorite woman who ranks first in his preference list. Each woman, who receives more than one proposals, rejects all but her most preferred man among them. Instead of marrying the man right now, the woman will be temporarily “engaged” to him.

At the second step, any man, who is rejected in the previous step, now proposes to his second choice. Each woman, who receives any proposal, chooses her favorite from the group consisting of the new “proposers” and any man she may have kept “engaged” at the preceding step.

\(^7\) We will talk more about the effects of introducing indifference to the stable marriage problem in section 2.3.3.
This procedure repeats itself in the same manner until no man is rejected. What is interesting about this algorithm is that upon its termination, everyone will be matched. Furthermore, these matched pairs actually constitute a stable matching, thereby proving to us that we can always find a stable matching for any stable marriage problem.

**Example 2.1: An illustration of Gale-Shapley Algorithm**

Consider the scenario given in Chapter 1. Let Andy be man 1, Benny be man 2, Canny be man 3, and Danny be man 4. Likewise, let Anna be woman 1, Betty be woman 2, Candy be woman 3 and Donna be woman 4. Thus, the new rank-ordered preference lists are as follows:

<table>
<thead>
<tr>
<th>Men's preferences lists</th>
<th>Women's preferences lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 3 4 1 2 4 3 1</td>
<td>1 2 4 3 1 2 4 3 1 2</td>
</tr>
<tr>
<td>2 2 4 1 3 2 4 3 1 2</td>
<td>3 1 3 2 4 1 2 4 3</td>
</tr>
<tr>
<td>3 4 1 3 2 3 1 3 2 4</td>
<td>4 1 2 4 3</td>
</tr>
<tr>
<td>4 2 1 3 4 4</td>
<td></td>
</tr>
</tbody>
</table>

The execution of the man-propose algorithm results in the following sequence of steps:

1. **Step 1:** Man 1 proposes to woman 1 (accepted)
   - Man 2 proposes to woman 2 (accepted)
   - Man 3 proposes to woman 4 (accepted)
   - Man 4 proposes to woman 2 (accepted, & woman 2 now rejects man 2)

2. **Step 2:** Man 2, now being “single”, proposes to woman 4 (accepted, & woman 4 now rejects man 3)

3. **Step 3:** Man 3, now being “single”, proposes to woman 1 (accepted, & woman 1 now rejects man 1)

4. **Step 4:** Man 1, now being “single”, proposes to woman 2 (rejected, for woman 2 prefers man 4 to man 1)

5. **Step 5:** Man 1 proposes to woman 3 (accepted)

Hence, the stable matching generated by the man-propose algorithm is \{ (man 1, woman 3), (man 2, woman 4), (man 3, woman 1), (man 4, woman 2) \}.

**2.3.2 Man and Woman Optimal Stable Matchings**

It would never cross one’s mind that all the men, who are essentially competing with each other for the women, can agree on a stable matching that is simultaneously optimal for all
of them. Likewise, the same thing can be said about the women. This is the case until Gale and Shapley made the following proposition in 1962.

Gale and Shapley (1962) discovered that the stable matching produced by the man-propose algorithm gives every man the best possible stable partner he can achieve in any stable matching. They called this matching the man-optimal stable matching. This is a remarkable result. It implies that if each man is independently given his best stable partner, then the result is a stable matching. Yet, there seems no a prior reason why this should even be a matching.

Likewise, if the roles of the sexes in the man-propose algorithm are reversed, then the resulting matching would be woman-optimal for all women. Every woman in this case will get her best stable partner.

As illustrated in Example 2.2, there are cases where the man-optimal stable matching is the same as the woman-optimal stable matching. However, in general, this is not true. Example 3 shows this property.

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**Example 2.2: Man-optimal stable matching is the same as woman-optimal stable matching.**

Using the same preference lists in Example 2.1, we can verify that the woman-optimal stable matching is the same as the man-optimal stable matching by running the woman-propose algorithm.

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8 The order in which the men propose is immaterial to the outcome. Even if we let Canny proposes first followed by Danny, Benny and finally Andy, the resulting matching is still the same. Therefore, to be exact, this is only one possible execution of the Gale-Shapley algorithm.
Example 2.3: Man-optimal stable matching is not the same as woman-optimal stable matching.

Consider the preference lists as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Man</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Woman</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

By applying the man-propose algorithm and the woman-propose algorithm, we see that:

- Man-optimal stable matching =\{(man 1, woman 1), (man 2, woman 2), (man 3, woman 3)\}
- Woman-optimal stable matching =\{(man 1, woman 3), (man 2, woman 1), (man 3, woman 2)\}

2.4 Structural Properties of Stable Marriage Problem

2.4.1 Conflict of Interests

McVitie and Wilson (1971) observed that while all the men will get their best stable partner in the man-optimal stable matching, each woman, on the contrary, will get her worst stable partner under the same matching. Hence, the stable matching that is termed as man-optimal is also woman-“pessimal”; likewise, woman-optimal is man-“pessimal”.

As pointed out later by Knuth (1976), this conflict of interest between the two sexes can be observed not only in comparing the optimal stable matchings for each sex, but also in comparing any stable matching.\(^9\) The inherent inference here is that when there are two stable matching, and given that a man m and a woman w are partners in one of these matching, it is impossible to have both man m and woman w to prefer the same stable matching.\(^10\)

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\(^9\) In real-life situations, the agents may not have the ability to distinguish among all alternatives facing them. There are many reasons for this, and perhaps the most important one is that the agents may have little information about some alternatives, and thus are indifferent between them.

\(^10\) This is only true when all preferences are strict. When indifferences are allowed, the woman may show equal liking for the two stable matchings despite the fact that she is matched to a different man in these two matchings.
Example 2.4: Conflict of interests between two sexes

From example 2.3, we see that in the man-optimal stable matching, all men get their first choice while the women, on the other hand, have their last choice. However, in the woman-optimal stable matching, all the women have their first choice while all the men get their last choice.

2.4.2 Weak-Pareto Optimality

Roth (1982) discovered that the man-optimal and the woman-optimal solution possess a further optimality property, sometimes referred to as the weak Pareto-optimality. The inherent implication behind this property is that there will not be any matching (stable or not) that can give all the men a mate who is better that the one they have under the man-optimal stable matching. By symmetry, an analogous result holds for the women and the woman-optimal stable matching.

2.4.3 “Fair” Stable Marriage Solution

Given the fact that the algorithm described in Gale and Shapley (1962) produces either a “man-optimal” or a “woman-optimal” stable outcome, depending on its implementation, there has been enormous search for a stable matching solution that is “fair” to both the men and the women. Several studies are reported to have found a stable matching that is “fair” to both sexes in some ways.

2.4.3.1 Egalitarian stable matching

Irving et al. (1989) introduced and studied the egalitarian stable marriage problem. They established that a stable marriage solution that minimizes the following objective function:

\[ \sum \sum (mr(m, w) + wr(w, m)) \]

\[ ^{11} \text{A simplified proof is found in Gale and Sotomayor (1985a).} \]
where \(mr(m, w)\) is the position of woman \(w\) in the man \(m\)’s list, and \(wr(w, m)\) is the position of man \(m\) in woman \(w\)’s list, can be found in polynomial time.

### 2.4.3.2 Minimum-regret stable matching

Suppose the ranking of an agent’s partner is a measure of his/her regret for the stable marriage solution. The minimum-regret problem is to find a stable marriage solution so that no single individual will be adversely affected (by having a high regret). Knuth (1976) (attributed to S.Selkow) designed an algorithm that can be used to solve the minimum-regret stable marriage problem.

### 2.4.3.3 Median stable matching

In their recent paper, Teo and Sethyraman (1997) introduced a new notion of “fairness” called the median stable marriage. They argued that it is fair for everyone to get his/her median partner. By listing out all possible stable partners and arranging them in order of decreasing preferences, an agent’s median partner is the one who is centrally positioned among the other stable partners. Suppose there are \(L\) stable partners, the agent’s median partner would be in the \((L+1)/2\) position. Teo and Sethyraman (1997) showed the surprising result that all the agents’ median partner actually constitute a stable matching.
Example 2.5: Median solution

In Example 2.3, there are altogether 3 stable matchings. The following list contains all the stable matchings for example 2.3, with the man-optimal stable matching listed first and the woman-optimal stable matching last.

1.) \{ (man 1, woman 1), (man 2, woman 2), (man 3, woman 3) \}

2.) \{ man 1, woman 2 ), (man 2, woman 3), (man 3, woman 1) \}

3.) \{ (man 1, woman 3), (man 2, woman 1), (man 3, woman 2) \}

By listing out all the possible stable partners for each agent and putting them in order of each agent’s decreasing preference, this is what we get:

<table>
<thead>
<tr>
<th>Man decreasing preferences</th>
<th>Woman decreasing preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 2 3</td>
<td>1 2 3</td>
</tr>
<tr>
<td>2 3 1</td>
<td>2 3 1</td>
</tr>
<tr>
<td>3 1 2</td>
<td>3 1 2</td>
</tr>
</tbody>
</table>

According to Teo & Sethyraman (1997), each agent’s median partner is in the \((1+3)/2 = 2\)nd position. Thus, the median stable marriage = \{ man 1, woman 2), (man 2, woman 3), (man 3, woman 1) \}.

2.5 Some Simple Extensions of Stable Marriage Problem

Many additional variants of the classical stable marriage problem have been introduced and studied. In this section, we consider the three common variants of the classical stable marriage problem.

2.5.1 Sets of Unequal Size

One simple extension of the basic problem is a problem with unequal number of men and women. McVitie and Wilson (1970) discovered that most of the results established for the basic problem could be applied, in a slightly amended form, to this variant.

When the man-propose algorithm is used, it will not terminate when everyone is “engaged” as in the classical problem. Instead, it will only end either when 1.) all women have been proposed to (if the number of men is greater than that of the women); or when 2.) every
man is either married or has been rejected by all the women (if the number of men is smaller than that of the women).

In either case, the resulting matching is stable, and it is again man-optimal and woman-pessimal in the usual sense. However, not everyone will be matched due to the unequal sizes in both sexes. In fact, those people, who are unmatched in this matching, will remain unmatched in all other stable matchings [Gale and Sotomayor (1985a)].

Example 2.6: An unmatched agent in a stable matching will always remain unmatched in other stable matchings.

<table>
<thead>
<tr>
<th>Men's preference lists</th>
<th>Woman's preference lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2</td>
<td>1 2 3 1</td>
</tr>
<tr>
<td>2 2 1</td>
<td>2 3 2 1</td>
</tr>
<tr>
<td>3 1 2</td>
<td></td>
</tr>
</tbody>
</table>

Consider following instance with 3 men and 2 women. Based on the above preference lists, there are only two stable matchings. The two stable matchings are as follows:

Man-optimal stable matching =\{(man 2, woman 2), (man 3, woman 1), man 1 unmatched\}

Woman-optimal stable matching =\{(man 2, woman 1), (man 3, woman 3), man 1 unmatched\}

Observe that in both stable matchings, man 1 remains unmatched.

2.5.2 Unacceptable Partners

In the basic marriage problem, it is assumed that everyone has a strict preference ordering over every member of the opposite sex. This assumption implies that everyone would strictly prefer to be married than to be single. However, in actual cases, a person may declare one or more of the opposite sex as unacceptable as he/she would rather remain single than to be married to someone he/she does not care. Consequently, the preference list of such a person contains only a proper subset of the members of the opposite sex. Such a preference list is termed as incomplete. We call this variant the stable marriage model with rejection.
Under this setting, a couple can be matched only if they are acceptable to each other. A stable matching is one in which 1.) all the matched couples are mutually acceptable, and 2.) there is no pair of man and woman, each of whom strictly prefers one another to his/her current partner.

Past studies [e.g. Gale & Sotomayor (1985a), Gusfield (1990)] show that the results concerning the basic problem can be applied to this variant. However, this does not imply that nothing new can be found in the context of unacceptable partners.

Gale & Sotomayor (1985a) discovered that the men and the women could each be partitioned into two sets-1.) those that have partners in all stable matchings and 2.) those that have no partners at all.

They further showed that when a man extends his list to include an previously unacceptable woman, no woman will be made worse off while no man, except the man himself possibly, will be made better off. This result holds true no matter which version of the Gale and Shapley algorithm is used.

### 2.5.3 Indifference

It is well known that every instance of the classical stable marriage problem admits at least one stable matching. In this basic problem, each person must rank the members of the opposite sex in strict order of preferences. However, in practical application, a person may not wish or be able to choose between alternatives, thus he may introduce ties into his preference list.

With the introduction of indifference, the notion of stability may be generalised in the three different ways suggested by Irving (1994). These three different notions are namely 1.)

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12 If there are n men and m women, with n>m, then n-m women will be unmatched. However, if n<m, then m-n men will be unmatched.
super-stable; 2.) strongly stable and 3.) weakly stable. Among these three notion of stability, the weakly stable matching is the weakest yet most reasonable one. Under this context, a matching is stable unless there is a man and a woman, each of whom strictly prefers the other party to his/her current partner.

It is immediate to see that if the ties are broken arbitrarily, any matching that is stable in the resulting (strict preferences) instance is weakly stable in the original instance (with indifferences).

As pointed out by both Gardeners (1975) and Irving (1994), a weakly stable matching can be found by breaking the ties arbitrarily and applying the Gale and Shapley algorithm. However, queries arise on how the ties should be broken. It may happen that different ways of breaking ties can result in different stable matchings.

Example 2.8: Different ways of breaking ties can result in different stable matchings
Consider the following preference lists in which all preferences are strict except for man 1 (indicated by the parentheses in man 1’s preference list)

<table>
<thead>
<tr>
<th>Men's preference lists</th>
<th>Woman's preference lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (2 3) 1</td>
<td>1 1 2 3</td>
</tr>
<tr>
<td>2 2 1 3</td>
<td>2 1 2 3</td>
</tr>
<tr>
<td>3 3 1 2</td>
<td>3 1 3 2</td>
</tr>
</tbody>
</table>

Method 1: Strict preference list of man 1 = 2, 3, 1
By employing the man-propose algorithm, the stable matching is {(man 1, woman 2), (man 2, woman 1), (man 3, woman 3)}.

Method 2: Strict preference list of man 1 = 3, 2, 1
By employing the man-propose algorithm, the stable matching is {(man 1, woman 3), (man 2, woman 2), (man 3, woman 1)}.
Observe that under these two different methods of breaking ties in man 1’s preference list, man 1 is not matched to a same partner despite the fact that the same matching mechanism is used.

2.6 Strategic Behavior of Agents

2.6.1 Impossibility of a “Strategy Proof” Stable Matching Mechanism

---

13 A matching mechanism is called “strategy-proof” if it makes it a dominant strategy for every agent to state his/her true preferences. So, a stable matching mechanism is called “strategy proof” if it produces stable matching, as well as it makes it a dominant strategy for all agents to state true preferences.
In the stable marriage problem, it is assumed that there is some sort of a matchmaker to do the matching. What everyone needs to do is to submit his/her strictly rank ordered preference list containing members of the opposite sex to the matchmaker.

Intuitively speaking, since there is no explicit rule saying that everyone should report his/her true preferences, some people may lie about his/her true preferences because lying may actually give him/her a better partner.

In fact, as pointed out by Roth (1982), lying can never be totally discouraged in the stable marriage problem. There will be cases where some agents find lying beneficial to them. The reason why some people may have the incentive to be deceitful is a direct consequence of the reality that it is impossible to design a stable matching mechanism that can make truth-telling a dominant strategy for everyone. In other words, there is no such thing as a “strategy-proof” stable matching mechanism.

2.6.2 Incentives Facing the Men when Man-propose Algorithm is used

Roth (1982) discovered that when the man-propose algorithm is used, every man would never have the incentive to lie about his preferences. This is because falsifying preferences will never result in giving him a stable partner who is strictly better than the one he gets when he reveals his true preferences.\(^\text{14}\)

Roth’s result is quite intuitive. With true preferences, each man would have already obtained his best stable partner (i.e. his man-optimal partner) when the man-propose algorithm is used. Thus, lying can never be beneficial to him.

\(^{14}\text{Likewise, when the woman-propose algorithm is used, every woman will have no incentive to falsify her preferences.}\)
Nevertheless, this is a remarkable result. It implies that when the man-propose algorithm is used, the incentives for lying can be totally precluded from the men, as truth telling is a dominant strategy for every man.

2.6.3 Incentives Facing the Women when Man-propose Algorithm is used

In their analysis of the strategic behavior of the women, Gale and Sotomayor (1985b) discovered that when the man-propose algorithm is used, some women would have the incentive to lie about their preferences. Hence, this indicates that lying can be beneficial to some women when the man-propose algorithm is used.

2.6.3.1 Optimal lying by the women

In fact, it turns out that each woman, by lying, can always force the man-propose algorithm to give her her woman-optimal stable partner even though the man-optimal (or rather the woman-pessimal) stable matching is computed. As suggested by Gale and Sotomayor (1985b), all she needs to do is to submit a preference list, which ranks all men in the same order as her true preference list but ranks as unacceptable those men that are inferior to her woman-optimal partner.\textsuperscript{15}

Undoubtedly, this is a remarkable lying strategy by Gale and Sotomayor (1985b). According to the way the man-propose algorithm assigns partners, each woman should get her worst stable partner when the man-propose algorithm is used. However, by just following Gale and Sotomayor (1985b)’s proposed strategy, a deceitful woman can actually escape from her fate.

\textsuperscript{15} Likewise, when the woman-propose algorithm is used, every man can force the woman-propose algorithm to give him his man-optimal stable partner though the woman-optimal stable matching is being computed.
of getting her worst stable partner. Better still, she can even obtain her woman-optimal stable partner.

**Example 2.9: An illustration of the optimal lying strategy proposed by Gale and Sotomayor (1985b)**

<table>
<thead>
<tr>
<th>Men's true preference lists</th>
<th>Women's true preference lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 1 2 1</td>
<td>2 1 2 1 2 1</td>
</tr>
</tbody>
</table>

Consider the following preferences list

By applying the man-propose algorithm and woman-propose algorithm respectively, we see that:

- Man-optimal stable matching = \{(man 1, woman 1), (man2, woman 2)\}
- Woman-optimal stable matching = \{(man 1, woman 2), (man 2, woman 1)\}

From the woman-optimal stable matching, we see that the best stable partner for woman 1 is man 2. Observe that woman 1 does not get man 2 under the man-optimal stable matching.

Suppose woman 1 follows Gale and Sotomayor’s lying strategy and declares man 1, who ranks below man 2, as unacceptable. Thus, the new preference lists are as follow:

<table>
<thead>
<tr>
<th>Men's preference lists</th>
<th>Women's preference lists</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 1 2 1 2 1</td>
<td>2 1 2 1 2 1</td>
</tr>
</tbody>
</table>

By applying the man-propose algorithm to these new preference lists, we see that

- Man-optimal stable matching = \{(man 1, woman 2), (man 2, woman 1)\}

Observe that woman 1 now gets man 2, who is her woman-optimal stable partner.

Note: Readers can verify that if woman 2 follows the same lying strategy, she can also get man 1, her best stable partner though the man-optimal stable matching is being computed.

Thus, when taken together, the above results shed some lights on how much misrepresentation can be precluded in any stable matching procedure. The first result by Roth (1982) shows that it is not possible to remove all incentives for misrepresentation, but his second result and that of Gale and Sotomayor (1985) indicate that misrepresentation can be removed from one of the two disjoint sets. By using either the man-propose algorithm or the woman-propose algorithm, the incentives for misrepresentation can be completely removed from either the men or the women. Last but not the least, the discovery by Gale and Sotomayor (1985b) indicates the maximum amount of benefits from misrepresentation.
2.7 Practical Applications of Gale and Shapley Algorithm

While writing their paper, Gale and Shapley (1962) had some reservations on whether the results found by them could be applied “in practice”. Surprisingly, not only could their ideas be applied, it had already been in use since 1952!

2.7.1 National Resident Matching Program

Introduced by the Association of American Medical College in 1952, the National Resident Matching Program (NRMP) is a centralised system used in the assigning of internships at hospitals to graduating medical students. Being a polygamous version of the Gale-Shapley algorithm, the matching procedure used produces a “hospital-optimal” stable matching. A look at the history of this internship labour market is significant enough to convince one the importance of the stability condition in a two-sided matching market. A modest attempt is made here to provide the readers an overview of this history. For more details, readers can refer to Roth (1984a, 1991).

Before 1952, appointments of the internship were done on the ad-hoc basis, which had resulted in enormous dissatisfaction to both the hospitals and students. Each hospital would compete with one another and make offers to the students independently, while the students were required to make their decisions within a certain deadline. Because of such an arrangement, problem began to arise and manifest itself in the waiting period between the time of an offer and the time of acceptance/rejection. For the students, having being offered internship by a less preferred hospital, they were unhappy to be pressured into accepting the offer, especially if a better offer came later. Thus, they were inclined to reply until the last minute, hoping to receive an offer from a preferred hospital. This kind of action had caused great difficulty to the hospitals.
The hospitals were unhappy with receiving a late rejection when the rest of their alternate candidates had accepted other offers. It would be of a greater dissatisfaction to them when a student, having accepted their offer, failed to fulfil his duties. Hence, this process had resulted in an “unstable” matching, and caused enormous dissatisfaction for both the hospitals and the students.

Attempts were made to solve the problem. These included reducing the waiting period. However, such efforts were fruitless.

It was until in 1950 that a centralized system replaced the previous competitive market. In the new process, each hospital would submit a rank ordered list of the students, while the students would each submit a rank ordered list of the hospitals. A matching would then be arranged based on these preferences. However, the earlier result obtained was unsatisfactory as it was found that the algorithm allowed some students to benefit by falsifying their preferences. Hence, the algorithm was modified to the present NRMP.

As NRMP is a voluntary process in which the participants are free to arrange their own matches outside the system without any compliance to the matching established by the system, the success of the system can be measured by its participation rate. The reality that the voluntary rate had always remained high signified the effectiveness of the Gale and Shapley algorithm in introducing a stability condition to the market.

However, it was recognized that the voluntary rate had dropped significantly starting from the 1970s. This great decline is attributed to the fact that the NRMP fails to cater to the needs of couples seeking for positions in the same city. Hence, this called for more modifications and improvements to be made to the system.
One of the latest issues concerning the NRMP is a decision to convert the present procedure from the “hospital-optimal” to the “student-optimal” matching procedure. For more details on this issue, refer to Roth (1997).

2.7.2 More Applications

In addition to the NRMP, similar ongoing programs are used by other professional groups and placement agencies. Examples include the General Medical Council in United Kingdom. What is common among these programs is that all are a voluntary system whereby the participants involved need not comply with the assigned matching produced by the system.

2.8 Hospitals/Residents Problem

Like the stable marriage problem, the hospitals/residents problem is made up of two discrete sets of agents. The two sets of agents are namely the residents and the hospitals, each of whom has a preference ordering for the members of the opposite set. What is different between these two problems is that in the hospitals/residents problem, each hospital has one or more places to fill (i.e. quota). Hence, a matching in the hospitals/residents problem is a many to one mapping between the residents and the hospitals so that no hospital exceeds its number of available places. In general, a matching in the hospital/residents problem is said to be unstable if there is a resident and a hospital, each prefers the other to his current partner (in the case of the hospital, to one of its assigned residents)
2.8.1 Existence of Stable Matching

In addition to the stable marriage problem, Gale and Shapley (1962) also considered the hospitals/residents problem in their paper. They observed that their original Gale and Shapley algorithm could always be modified to solve the hospitals/residents problem. This observation is significant as it implies that the existence of stable matching as defined in the marriage problem could be carried over to the hospitals/residents problem.

2.8.2 Hospital and Resident Optimal Stable Matching

Gale and Shapley (1962) further showed that the existence of optimal stable matchings as defined in the marriage problem could be carried over to the hospitals/residents problem. In other words, there exists a hospital-optimal stable matching that gives every hospital its best stable partners. As discovered by Gale and Shapley (1962), a polygamous version of their original algorithm, which is commonly known as the hospital-propose algorithm, can be used to find the hospital-optimal stable matching. Similarly, there also exists a resident-optimal stable matching that gives every resident his best stable hospital. This matching can be found using what we call the resident-propose algorithm.

2.8.3 Conflict of Interests

The kind of conflicts of interests that is being observed in the marriage problem can also be found in the hospitals/residents problem. It was found in Gusfield (1990) that while all the hospitals get their best stable partners in the hospital-optimal stable matching, each resident in turn has the worst stable partner. Hence, the stable matching that is termed as hospital-optimal is also resident-“pessimal”; likewise, resident-optimal is hospital-pessimal.
2.8.4 Weak-Pareto Optimality

Likewise, the weakly Pareto optimality still holds in the resident-optimal stable matching. However, the same cannot be said about the hospital-optimal stable matching. Roth (1985) discovered that when hospitals have responsive preferences, there might be matchings that all hospitals strictly prefer to the hospital-optimal stable matching. This implies that the hospital-optimal stable matching does not possess the weak Pareto optimality property as defined for the man and the woman-optimal stable matchings.

Example 2.10: There exists a matching that all hospitals strictly prefer to the hospital-optimal stable matching.

Consider the following preferences lists, in which hospital 1 has 1 position, hospital 2 has 1 positions, and hospital 3 has 2 positions.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
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<td>4</td>
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<th>1</th>
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<td>1</td>
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<td>2</td>
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<td>3</td>
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<td>3</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Hospitals' preferences | Residents' preferences

By applying the hospital-propose algorithm, we see that the hospital-optimal stable matching $h^* = \{(\text{hospital 1, resident 1}), (\text{hospital 2, resident 2}), (\text{hospital 3, residents 3 and 4})\}$

Now, consider the following matching, called $y = \{(\text{hospital 1, resident 3}), (\text{hospital 2, resident 1}), (\text{hospital 3, residents 2 and 4})\}$.

Observe that in $y$, both hospitals 1 and 2 gets its first choice resident respectively. So, we can conclude that both hospital 1 and 2 strictly prefer $y$ to $h^*$.

Likewise, with responsive preferences, hospital 3 also strictly prefers $y$ to $h^*$ since in $y$, it is assigned its second and fourth choice residents. This is a better matching compared to $h^*$ that gives it its third and fourth choice residents. Thus, $y$ is strictly preferred by all hospitals to $h^*$.

---

16 Gale and Shapley (1962) used the terminology “college admissions problem”
17 A hospital is said to have responsive preferences if for any two matchings that differ in only one resident, it prefers the matching containing the more preferred resident. For example, suppose there are two matchings, called $x$ and $y$ respectively. Matching $x$ assigns the hospital its first and third choice residents, while matching $y$ assigns the hospital its first and second choice residents. The hospital would prefer $y$ to $x$ if its preferences were said to be responsive.
2.8.5 Strategic Behavior of Agents

2.8.5.1 Impossibility of “strategy-proof” stable mechanism

Likewise, it is also impossible to design a stable matching mechanism that will make it a dominant strategy for all agents in the hospitals/residents problem to state their true preferences. Roth (1985), who is also the one who notes the impossibility of having a “strategy-proof” stable mechanism in the stable marriage problem, makes this discovery.

2.8.5.2 Incentives for the hospitals

Roth (1985) discovered that there is no stable matching procedure that can make it a dominant strategy for all hospitals to state their true preferences when the hospitals have responsive preferences. This implies that even when the hospital-propose mechanism (or any other stable matching mechanism that produces hospital-optimal stable matching) is used, some hospitals will have the incentives to falsify their preferences.

Example 2.11: It is not a dominant strategy for all hospitals to state their true preferences when the hospital-propose algorithm is used.

From example 2.10, we see that the hospital-optimal stable matching \( h^* = \{(\text{hospital 1, resident 1}), (\text{hospital 2, resident 2}), (\text{hospital 3, residents 3 and 4})\} \)

Now, suppose hospital 3 lies, and reports a preference ordering of 2, 4, 1, 3.

Applying the hospital-propose algorithm, the new hospital-optimal stable matching \( h^{**} = \{(\text{hospital 1, resident 3}), (\text{hospital 2, resident 1}), (\text{hospital 3, residents 2 and 4})\} \).

Observe that in \( h^{**} \), hospital 3 is assigned to its second and fourth choice residents. For hospital 3, this matching is clearly better than \( h^* \) that gives it its third and fourth choice residents. Thus, hospital 1 would have an incentive to lie about its preferences.

2.8.5.3 Incentives for the residents

On the other hand, Roth (1985) found out that it is still a dominant strategy for all residents to tell the truth when the resident-propose algorithm (or any other stable matching mechanism that yields the resident-optimal stable matching) is used. However, when the
hospital-propose algorithm is used, some residents will have the incentives to falsify his/her preferences.
3 STRATEGIC ISSUES IN GALE-SHAPLEY STABLE MARRIAGE GAME

3.1 Introduction

In the context of the stable marriage problem, numerous researches have been conducted to investigate the incentives each agent has in falsifying his/her preferences. Specifically, these studies seek to understand what can be achieved if an agent is deceitful and does not report his/her true preferences. In the theoretical literature, there are many ways to model this possibility. One common way is to assume that the deceitful agent knows about the true preferences lists of the other agents and such lists are to be reported truthfully. Under this assumption, numerous results, which have important implications for economics and game theory, have been established.

Roth (1982) found out that when the man-propose algorithm is used, every man would never be better off by falsifying his preferences. Hence, we can conclude that every man will never have the incentive to lie about his preferences when the man-propose algorithm is used.

However, the same conclusion cannot be generalized to every woman. Under the man-propose mechanism with the assumption that women can declare certain men to be unacceptable, Gale and Sotomayor (1985b) discovered that every woman will always have the incentive to falsify their preference lists as long as she does not obtain her best stable partner (i.e. her woman-optimal partner). It turns up that each woman can always get her woman-optimal stable partner from cheating. According to Gale and Sotomayor (1985b), all she needs to do is to state a preference list that ranks the men in the same orders as in her true preference list but ranks as unacceptable all men who are ranked below her woman-optimal stable partner.

Undoubtedly, this is a remarkable lying strategy for the women to adopt. By just declaring some men as unacceptable, every woman can always force the man-propose algorithm
to give her her woman-optimal stable partner although the man-optimal (or rather the woman-pessimal) stable matching is computed. However, prior to our study, we know of no analogous general results when the women are required to submit a complete preference list, i.e. the women cannot opt to remain single in the matching game. Interestingly, this restriction, which is the original model studied by Gale and Shapley, changes the complexity of the problem, as it is not known how a woman should lie by permuting her preferences under the man-propose algorithm. Nevertheless, from an example constructed by Josh Benaloh [cf. Gusfield and Irving (1989)], we know that lying by permuting preferences could result in the woman getting her woman-optimal stable partner from the man-propose algorithm.

These observations raise several intriguing questions, some of which are: Suppose the agents are only permitted to submit a complete preference list i.e. they are only allowed to lie by permuting their true preference lists. Can we identify instances in which an agent can benefit by falsifying his (her) preferences? If so, what is the ‘optimal’ cheating strategy for such an agent? Can a woman always force the man-propose matching mechanism to return her her woman-optimal partner?

In this chapter, we address these issues. Our proposed algorithm is in a sense optimal, as it gives the deceitful agent the best possible stable partner he (she) can have under the Gale-Shapley model.

For simplicity of presentation, we assume that the man-propose mechanism is used. We express our algorithm in a form that is meant for the women to use since stating true preferences is a dominant strategy for the men under the man-propose algorithm.\(^\text{18}\)

\(^{18}\) We can easily convert our proposed heuristic for the men to use by just reversing the sex roles.
3.2 Organization

In section 3.3, we shall define the formal model used in our study. That section also incorporates the common notations used and basic assumptions made in our study. Our main results appear in section 3.4, where we analyze the optimal strategy of a woman in the Gale-Shapley model. In that section, we develop our cheating strategy. We prove that the solution provided by our strategy is optimal in the sense that it gives every woman the best possible stable partner she can get when she is only allowed to lie by permuting her preferences. A numerical example will be given at the end of this section to illustrate how our cheating heuristic works, and to show that it is not always possible for a deceitful woman to recover her woman-optimal stable partner given that she can only cheat by permuting her preferences. Finally, in section 3.5, we study the implications of this algorithm for various strategic issues in the Gale-Shapley stable marriage game.

3.3 Formal Model and Basic Assumptions

3.3.1 Formal Model

For the sake of simplicity in presentation, we adopt the model of the classical stable marriage problem. There are two disjoint sets of size n, namely the men (M) and the women (W), where M={m_1, m_2, ..., m_n} and W={w_1, w_2, ..., w_n}.

Associated with each person is a strictly ordered preference list containing all the members of the opposite sex. Person x is said to prefer y to z if y precedes z on x’s preference list. We write y >_x z to denote that person x prefers y to z.

---

19 Since we do not allow any agent to declare anyone as unacceptable, it is important that each agent’s preference list should contain every member of the opposite sex.
A matching $\mu$ is a one to one mapping between the two sexes. The matching $\mu$ is said to be unstable if there is a man $m_i$, and a woman $w_i$, who are not matched in $\mu$ but each prefers to be matched with each other. Man $m_i$ and woman $w_i$ is said to block $\mu$ and is known as a blocking pair for the matching. A matching for which there is no blocking pair is said to be stable.

### 3.3.2 Basic Assumptions

There is complete information about the true preferences of all agents. When a deceitful woman lies about her preferences, she cheats under the assumption that others agents state their true preferences. We also assume that the man-propose algorithm is used. Hence, there is no incentive for the men to cheat.

### 3.4 Optimal Lying Via Permuting Preferences

To provide the intuition for our strategy, let us first look at how a woman, say $w$, can determine her woman-optimal stable partner under the man-propose mechanism. Surprisingly, we can show that $w$ can actually determine her woman-optimal stable partner without knowing the true preferences of all other agents involved. *What she needs to do is simply find all the men who will end up with a partner whom they consider inferior to her if she decides to remain single.* It turns out that the best among these men is actually her woman-optimal stable partner. Building on this insight, we develop an optimal strategy for $w$ when she is only allowed to cheat by permuting her preferences. By repeatedly calling the man-propose algorithm (i.e. Gale-Shapley algorithm), and using our knowledge of the men who have proposed to $w$ in the algorithm, we can easily construct the optimal strategy. This strategy is said to be optimal because it can help woman $w$ to optimize the partner she can get when she can only cheat by
permuting her preferences. To conclude this section, we provide a numerical example to illustrate how our cheating heuristic works.

### 3.4.1 Finding Woman-optimal Stable Partner using the Man-propose Algorithm

Let us consider a matching game with \( n \) equal number of men and women, each of whom is stating his/her true preferences. Given these preferences, how do we determine the best stable partner (i.e. woman-optimal stable partner) for a particular woman, say \( w \)?

With complete information about all agents’ preferences, we can easily determine which man is \( w \)’s woman-optimal stable partner. All we need to do is to run the woman-propose algorithm.

Now, suppose we are to determine \( w \)’s woman optimal stable partner using the man-propose algorithm, how should we solve the problem?

We came up with a way to find the woman-optimal stable partner for \( w \) using the man-propose algorithm. The following is a description of how our procedure works.

<table>
<thead>
<tr>
<th>A procedure to find woman-optimal stable partner using man-propose algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Modify ( w )’s preference list by inserting the option to remain single at the beginning of her list.</td>
</tr>
<tr>
<td>2. Run the man-propose algorithm. Due to the modification made to ( w )’s preference lists, we would see that all proposals made to ( w ) are being rejected. At the end of the algorithm, ( w ) and a man, say ( m ) will remain single.</td>
</tr>
<tr>
<td>3. Look at all the men who have proposed to ( w ) in stage 2. The best among them, denoted by ( m^1 ), is ( w )’s woman-optimal partner.</td>
</tr>
</tbody>
</table>

**Theorem 1:** \( m^1 \) is \( w \)’s woman-optimal partner
Proof:

Let $m(w)$ denote $w$’s woman-optimal stable partner. Let us modify $w$’s preference list by inserting the option to remain single in the list, immediately after $m(w)$. In other words, we declare all men that are inferior to $m(w)$ as unacceptable. Consequently, in the man-propose algorithm, all proposals inferior to $m(w)$ will be rejected.

Since there exists a stable matching with $w$ matched to $m(w)$, our modification to $w$’s preference list would not destroy this solution. The man-optimal matching for this modified preference list must match $w$ to $m(w)$. This implies that $m(w)$ must have proposed to $w$ during the execution of the man-propose algorithm.

Note that until the time when $m(w)$ proposes to $w$, the man-propose algorithm for the modified list runs exactly in the same way as in stage 2 of our proposed procedure. The difference is that in stage 2, the proposal from $m(w)$ would be rejected whereas the man-propose algorithm for the modified list will accept the proposal from $m(w)$ as $m(w)$ is preferred by $w$ to the option of being single. Hence, we can conclude that $m(w)$ is among those men who have proposed to $w$ in stage 2.

Suppose $m^1 >_w m(w)$, i.e. $m^1$ is preferred by $w$ to $m(w)$. Now, let’s consider the modified list, but this time the option of remaining single is placed immediately after $m^1$. Run the man-propose algorithm with this modified list. Up until the time $m^1$ proposes to $w$, the algorithm runs in exactly the same way as in stage 2. After which, the algorithm will return a stable partner for $w$ who is at least as good as $m^1$. This contradicts with our earlier assumption that $m(w)$ is $w$’s woman-optimal stable partner. Hence, $m^1$ must be the same man as $m(w)$, i.e. $m^1 = m(w)$. ♦
So, in order to determine her woman-optimal stable partner, what woman w needs to do is to simply find out who will be disappointed if she opts to remain single throughout the matching process. This is because, as we have shown in the proof, the most preferred man in this set of disappointed men\(^{20}\) is her woman-optimal stable partner.

Note that, in our procedure to determine the woman-optimal stable partner for w, the true preference list for w is only used to compare those men who have proposed to her. We do not need to know the exact preference list of w. We only need to know which man is the best among a given set of men, according to w. Hence, the information needed to find the woman-optimal partner of w is much less than what is required when the woman-propose algorithm is used. This is useful for the construction of our cheating strategy for w as the information on the preference list for w is not given apriori and needs to be determined.

### 3.4.2 Cheating Your Way to a Better Marriage

Observe that the preceding procedure only works when woman w is allowed to remain single throughout the matching process so that she can reject any proposal made to her in the algorithm. In other words, she is given the option to declare any man as unacceptable.

Suppose, we do not give w an option to declare any man as unacceptable, then how should w determine her best stable partner? Note that this is essentially the same as determining the best stable partner w can get from the man-propose algorithm when she can only cheat by permuting her preferences.

A natural extension of the strategy is for w to

(i) Accept a proposal first, and then reject all future proposals.

---

\(^{20}\) These men are disappointed because they will be matched with a less desirable partner.
(ii) Find her most preferred partner from the list of men who have proposed to her but are rejected by her earlier on. Repeat the man-propose algorithm until this man proposes to her.

(iii) Reverse her earlier decision and accept the proposal from this man. Continue the man-propose algorithm by rejecting all future proposals.

(iv) Repeat (ii) and (iii) until she cannot find a better partner from all other proposals.

Unfortunately, this elegant strategy does not always yield the best stable partner w can achieve via cheating. The main reason is that this greedy improvement technique does not allow w to reject her current best possible partner, in the hope that this rejection will trigger a proposal from a better would-be partner.

Before we provide a detailed description of our heuristic, let us first introduce some common notations, which we will be using in the sequel.

Let

\[ P(w) = \text{the true preference list of woman } w, \text{ where } P(w) = \{m_1, m_2, m_n\}, \]

\[ P(m, w) = \text{a preference list for } w \text{ that returns } m \text{ as the man-optimal partner for } w. \]

Our heuristic will construct \( P(m, w) \) iteratively

<table>
<thead>
<tr>
<th>Generally, our proposed heuristics comprises of the following steps:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Run the man-propose algorithm with the true preference list ( P(w) ) for woman ( w ). Keep track of all the men, who have proposed to ( w ). Suppose the man-optimal partner for ( w ) is ( m ). Let ( P(m, w) ) be the true preference list ( P(w) ).</td>
</tr>
<tr>
<td>2. Suppose ( m_i ) is one of the men (excluding ( m )) who have proposed to ( w ) in stage 1. By moving ( m_i )</td>
</tr>
</tbody>
</table>
to the front of the list of $P(m, w)$, we obtain a preference list for $w$ such that the man-optimal partner will be $m_j$. Let $P(m_j, w)$ represents this altered list. We say that $m_j$ is a possible partner for $w$.

3. Repeat step 2 for every other man (except $m$) who has proposed to woman $w$ in stage 1. After which, we say that we have already exhausted man $m$, the man-optimal partner obtained with the preference list $P(m, w)$.

4. Suppose there are some man $m_k$ who is a possible partner for $w$ but has not been exhausted, run the man-propose algorithm with $P(m_k, w)$ being the preference list of $w$. Identify new possible partners and their associated preference lists. Then, classify $m_k$ as exhausted.

5. Repeat Step 4 until all possible partners of $w$ are exhausted. Let $N$ denote the set of all the exhausted partners for $w$.

6. Among all the men in $N$, identify the man (say $m_a$) whom woman $w$ preferred most. We conclude that $m_a$ is the best “man-optimal” stable partner for $w$ when she is only allowed to lie by permuting her preferences. This implies that $P(m_a, w)$ is the optimal cheating strategy for $w$.

The men in the set $N$ at the end of the algorithm have the following crucial properties:

i.) For each man $m$ in $N$, there is an associated preference list for $w$, denoted by $P(m, w)$, which can result in $w$ getting $m$ as her man-optimal partner.

(ii) For each $P(m, w)$ submitted, all proposals must come from the men in $N$ (otherwise, there will be some possible partners who are not exhausted).

With each run of the Gale-Shapley algorithm, we exhaust a possible partner. Hence, this algorithm executes at most $n$ Gale-Shapley algorithms before termination.
Theorem 2: $\pi = P(m_p, w)$ is an optimal cheating strategy for woman $w$.

Note: We use the convention that $r(m) = k$ if man $m$ is the $k^{th}$ man on woman $w$'s list.

Proof:

Let $\pi^* = \{m_{p1}, m_{p2}, \ldots, m_{pn}\}$ be the preference list that gives rise to the best stable partner for $w$ under the man-propose algorithm. Let this man be denoted by $m_{pq}$, whom woman $w$ strictly prefers to $m_a$ under true preferences. Recall that $r(m_i)$ represents the rank ordering of $m_i$ under the true preferences of $w$. Hence, we can conclude that $r(m_{pq}) < r(m_a)$.

Observe that for those men who never get to propose to woman $w$, their orders on the list of woman $w$ does not affect the outcome of the man-propose algorithm. Furthermore, all the men with a rank ordering smaller than that of $m_{pq}$ do not get to propose to $w$, otherwise we can cheat further to improve on the partner for $w$, contradicting the optimality of $\pi^*$. Thus, we can arbitrarily alter the orders of these men. Without loss of generality, we may assume that:

$$1 = r(m_{p1}) < 2 = r(m_{p2}) < \ldots < q = r(m_{pq})$$

Since $r(m_{pq}) < r(m_a)$, we should find $m_{pq}$ preceding $m_a$ in $\pi^*$ or else $\pi^*$ would give to $m_a$. So, $\pi^*$ can be visualized as taking the form $\{m_{q1}, m_{q2}, \ldots, m_{qn}\}$ where $m_a$ can take any position from $m_{p,q+1}$ to $m_{pn}$.

Now, modify $\pi^*$ such that all men who rank higher than $m_a$ but lower than $m_{pq}$ (under true preferences) are put in accordance to their ranks. This is accomplished by moving all these men before $m_a$ in $\pi^*$. With that alteration, we obtain a new preference list $\pi' = \{m_{q1}, \ldots, m_{qn}\}$ such that

(i) $$1 = r(m_{q1}) < 2 = r(m_{q2}) < \ldots < s = r(m_{qs})$$

(ii) $$m_{q1} = m_{p1}, \ldots, m_{q1} = m_{pq},$$

where the position of those men who rank higher than $m_{pq}$ is unchanged,
(iii) $r(m_a) = s+1$, $m_a \in \{m_{qs+1}, \ldots, m_{qn}\}$, and

(iv) each man in the set $\{m_{qs+1}, \ldots, m_{qn}\}$ retains his position relative to one another under $\pi^*$. 

Note that the man-optimal partner of $w$ under $\pi^1$ cannot come from the set $\{m_{qs+1}, \ldots, m_{qn}\}$. Otherwise, the same partner would be obtained under $\pi^*$ since each man in the set $\{m_{qs+1}, \ldots, m_{qn}\}$ retains his position relative to one another under $\pi^*$. This leads to a contradiction as $\pi^*$ is supposed to return a much better partner for $w$. Since the man-optimal partner of $w$ under $\pi^1$ cannot come from the set $\{m_{qs+1}, \ldots, m_{qn}\}$, this implies that $\pi^1$ returns a better partner compared to that under $\pi$. In fact, $\pi^1$ is another permutation that will match $w$ to $m_{pq}$.

Now, since the preference list $\pi$ returns $m_a$ with $r(m_a) = s+1$ in $\pi$, we may conclude that the set $N = \{m_a, m_{i1}, \ldots, m_{ip}\}$ (obtained from the final stage of the algorithm) does not contain any man of rank smaller than $s+1$. Thus $N \subseteq \{m_{qs+1}, \ldots, m_{qn}\}$. Suppose $m_{qs+1}, m_{qs+2}, \ldots, m_{qw}$ do not belong to the set $N$, and $m_{qw+1}$ is the first man after $m_{qs}$ who belongs to the set $N$. By the construction of $N$, there exists a permutation $\pi^{11}$ that returns $m_{qw+1}$ as the man-optimal stable partner for $w$ under the man-propose algorithm. Furthermore, since all the men who propose to $w$ in the course of the algorithm are in $N$, they are no better than $m_a$ to $w$. Hence all proposals must come from the men in $\{m_{qw+1}, \ldots, m_{qn}\}$.

By altering the orders of those men who do not propose to $w$, we may assume that $\pi^{11}$ takes the following form: $\{m_{q1}, \ldots, m_{qp-1}, m_{qs}, \ldots, m_{qw}, m_{qw+1}, \ldots\}$, where the first $qw+1$ men in the list are identical to that in $\pi^1$. But, the man-optimal stable solution obtained using $\pi^{11}$ must also be stable under $\pi^1$, since $w$ is match to $m_{qw+1}$, and the set of men she strictly prefers to $m_{qw+1}$ is
identical in both $\pi^{11}$ and $\pi^1$. This is a contradiction as $\pi^1$ is supposed to return a man-optimal solution better than $m_a$.

This will imply that $\pi^*$ does not exist. Hence, we can conclude that $\pi$ is optimum and $m_a$ is the best man-optimal stable partner $w$ can get by permuting her preferences. ♦

### 3.4.3 A Numerical Example

After seeing how we have developed our cheating strategy, let us now take a look at how we can apply it. The following numerical example illustrates how our strategy works.

**Example 3.1: Numerical example to illustrate how our heuristic works.**

Consider the preference lists as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>4</td>
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<td>2</td>
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<td>1</td>
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<td>1</td>
<td>4</td>
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<td>3</td>
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<td>3</td>
<td>5</td>
<td>1</td>
<td>4</td>
<td>2</td>
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<td>4</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Men's true preferences | Women's true preferences

By applying the man-propose and woman propose algorithm, we see that:

Man-optimal stable matching = \{ (man 1, woman 2), (man 2, woman 3), (man 3, woman 5), (man 4, woman 4), (man 5, woman 1) \}, with woman 1 getting her fourth choice, man 5.

Woman-optimal stable matching = \{ (man 1, woman 1), (man 2, woman 2), (man 3, woman 3), (man 4, woman 4), (man 5, woman 5) \}, with woman 1 getting her first choice, man 1.

Let us construct the optimal cheating strategy for woman 1.

**Step 1:** Run the man-propose algorithm with the true preference list for woman 1. The
man-optimal partner for $w$ is man 5. Man 4 is the other man who proposes to
woman 1 in the algorithm. $P(\text{man 5, woman 1})=(1,2,3,5,4)$.

<table>
<thead>
<tr>
<th>Step 2-3:</th>
<th>Man 4 is moved to the head of woman 1’s preference list. i.e. $P(\text{man 4, woman 1})=(4,1,2,3,5)$. Man 5 is exhausted and man 4 is a possible partner for $w$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 4:</td>
<td>As man 4 is not yet exhausted, we run the man-propose algorithm with $P(\text{man 4, woman 1})$ as the preference list for woman 1. Man 4 will be exhausted after this, and man 3 is identified as a new possible partner for $w$. $P(\text{man 3, woman 1})=(3,4,1,2,5)$.</td>
</tr>
<tr>
<td>Repeat Step 4:</td>
<td>As man 3 is not yet exhausted, we run the man-propose algorithm with $P(\text{man 3, woman 1})$ as the preference list for woman 1. Man 3 will be exhausted after this. No new possible partner is found, and so the heuristic terminates.</td>
</tr>
</tbody>
</table>

From the above example, we see that woman 1 should cheat in order to get a partner better than her original man-optimal partner, man 5 (her true fourth choice man). However, it turns out that the best stable partner she can achieve by permuting her preferences is man 3 (her third choice man) instead of man 1 (her woman-optimal stable partner). Hence, we can see that, unlike in the case where declaration of partners (i.e. rejection) is allowed in the cheating strategy, woman 1 cannot always assure herself of getting her woman-optimal partner by permuting her preferences.

### 3.5 Strategic Issues in Gale-Shapley Game

By requiring the women to submit complete preference lists, we are clearly restricting their strategic options, and thus many of the strong structural results known for the model with
rejection (i.e. declaration of unacceptable partners is allowed) may not hold in this model. This is, in a sense, good as it reduces the possibility that the women will cheat. In the rest of this section, we present several examples to show that the strategic behavior of the women can be very different under the model with and without rejection.

3.5.1 The best possible stable partner needs not be the woman-optimal stable partner

In the two-sided matching model with rejection, it is not difficult to see that the women can always force the man-propose algorithm to return the woman-optimal solution. To get her woman-optimal partner from the man-propose algorithm, each woman just needs to reject all those men who are inferior to her women-optimal partner.

In our model, where rejection is forbidden, we have seen earlier in Example 3.1 that woman 1 cannot force the man-propose mechanism to return her the woman-optimal partner. In fact, this is the case even if all of them collude. A simple example to illustrate this is when each man has his own distinct first choice woman (i.e., the first ranked woman is different for every man). In this case, there is no conflict among the men. The resulting man-optimal stable matching will always give each man his true first choice woman regardless of how the women rank the men in their lists. Hence, by ruling out the strategic option of remaining single for the women, it significantly affects their ability to change the outcome of the game by cheating.

Furthermore, by repeating the above analysis for all the other women in Example 4.1, we can conclude that the best possible stable partner for woman 1, 2, 3, 4, and 5 are respectively man 3, 1, 2, 4, and 3. An interesting observation here is that woman 5 cannot benefit by cheating alone. No matter how she cheats, woman 5 still gets back her original man-optimal partner i.e.
man 3. However, when woman 1 cheats using the preference list (3,4,1,2,5), woman 5 will also benefit as she will be matched to man 5, her true first choice man.

3.5.2 Multiple Strategic Equilibria

Suppose each woman w announces a preference list of $\pi(w)$. The set of strategies \{\pi(1), \pi(2), \ldots, \pi(n)\} is said to be in strategic equilibrium if none of the women has an incentive to deviate unilaterally from this announced strategy.

It is easy to see that if a woman benefits from announcing a different permutation list (instead of her true preference list), then every other woman would not be worse off.

**Theorem 3:** In general, if a single woman can benefit by cheating, then the game has multiple strategic equilibria.

**Proof:**

A strategic equilibrium can be constructed by repeating our cheating algorithm iteratively, improving the partner for some woman at each iteration. Notice that the partner of a woman at the end of iteration j is at least as good as her partner at the beginning of the iteration j. The algorithm will thus terminate at a strategic equilibrium, where at least one woman will be matched to someone whom she (strictly) prefers to her man-optimal partner.

Another strategic equilibrium is obtained when each woman w announces a list of the form \{w (1), w (2), \ldots, w (n)\}, with w (1) being the man-optimal partner and w(2),\ldots,w(n) in the same order as in the true preference list. In this case, the man-propose algorithm will match woman w to w (1) since moving w (1) to the front of w’s preference list does not affect the sequence of proposals in the man-propose algorithm. No woman can benefit from cheating, as all other
women are already matched to their announced first-ranked partner. Thus, we have constructed two strategic equilibria. ♦

3.5.3 Does It Pay to Cheat?

Roth (1982) showed that under the man-propose mechanism, the men would have no incentives to alter their true preference lists. However, in the rejection model, Gale and Sotomayor (1985a) showed that a woman would always have an incentive to cheat as long as she has at least two distinct stable partners. Bittel (1989) showed that the average number of stable solution is asymptotic to nlog (n)/e, and with high probability, the rank of the woman-optimal and man-optimal partner for the woman are log (n) and n/log (n) respectively. Thus, in typical instances of the stable marriage game under the rejection model, most of the women will not reveal their true preference lists. This raises interesting questions as to whether there is a matching mechanism that can induce all men and women to reveal their true preferences. Unfortunately, this is ruled out by the famous result of Roth (1982) which shows that there is no such thing as a strategy-proof stable matching mechanism. Hence, it is not difficult to realize that in the rejection model, information about the preferences of other agents is important to a typical woman as she can then exploit the system with such information.

Many researchers have argued that the troubling implications from these studies are not relevant in practical stable marriage game, as the rejection model assumes that the women have full knowledge of each individual's preference list and the set of all participants in the game. For the model we consider, it is natural to ask whether it pays a typical woman to solicit information about the preferences of all other participants in the game. The question we would like to address is whether she can benefits from cheating even if she has these information. We run the proposed
cheating strategy on 1000 instances, generated uniformly at random, for \( n = 8 \). The number of women who benefit from cheating is tabulated in Table 3.1.

Table 3.1 Benefit from Cheating: 8 by 8 case

<table>
<thead>
<tr>
<th>Number of women who benefit</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of observations</td>
<td>740</td>
<td>151</td>
<td>82</td>
<td>19</td>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

As shown in Table 3.1, the number of women who can gain from cheating actually reduces drastically. In 74% of the instances, the women realize that the man-optimal solution is their only option, no matter how they cheat. The average percentage of women who benefit from cheating is merely 5.06%.

To look at the typical strategic behavior on larger instances of the stable marriage problem, with the help of V.J. Sethuraman of MIT, we run the heuristic on 1000 random instances for \( n = 100 \). The observations are summarized in Figure 3.1. The cumulative plot shows a much nicer pattern (Figure 3.2). In particular, in more than 60% of the instances, there are at most 10 women (out of 100) who benefit from cheating; in more than 96% of the instances, there are at most 20 women benefit from cheating. The average number of women who benefit from cheating is 9.515%.

Thus, the chances that a typical woman can benefit from obtaining complete information of the preferences of the other participants are pretty slim in our model.

A practical advice for all women in the stable marriage game, with man-optimal matching mechanism: "*don't bother to cheat!*"
Figure 3.1 Benefits from Cheating: 100 by 100 case

Figure 3.2 Benefits from Cheating: 100 by 100 case (Cumulative Plot)
4 STUDY OF TWO-SIDED MATCHING MARKET IN SINGAPORE

4.1 Introduction

Various forms of two-sided matching markets can be found in Singapore. One interesting situation that is worth studying is the primary six pupils/secondary schools market, which assigns primary six pupils to secondary schools upon the pupils' completion of their Primary School Leaving Examination (PSLE).

Like most of the markets (e.g. the U.K internship market and the U.S internship market) that we have come across so far, the primary six pupils/secondary schools market is centrally organized. Prior to taking his PSLE, each pupil would submit his rank ordering of schools to the Ministry of Education (MOE), which would later use this information to match the pupils to schools on the release of the PSLE results. However, unlike other centralized markets, this is a market where rejection is not allowed and it is compulsory that every pupil is matched to a school during the posting process.

Undoubtedly, the Ministry of Education (MOE) has done an excellent job in ensuring that the primary six pupils are assigned to the secondary schools in the fairest possible manner. Interestingly, the posting method used by the MOE is actually a form of student-propose algorithm. If the current algorithm produces stable solution, then according to Roth (1982), it is a dominant strategy for each pupil to reveal his/her true preferences. However, in reality, this may not be the case where some pupils may actually be induced not to reveal their preferences. Why is it so? This is one issue that this research seeks to elucidate. In this chapter, a modest attempt is made to study the algorithm used in the placement of primary six pupils in secondary schools. The main objective is to understand the behavior of both pupils and schools, identifying the problems the current posting system pose for them.
4.2 Sources of Information

Information regarding the current posting method is obtained from the Ministry of Education through an interview with the relevant personnel. Supplementary information is obtained from the Ministry of Education (MOE)’s Web site: http://www.moe.edu.sg and press clippings.

4.3 Organization

In section 4.4, we briefly describe the current secondary school education system in Singapore. Details about the current posting method used in the placement of primary six pupils to secondary schools are provided in section 4.5. Section 4.6 incorporates the formal model used in our study. In section 4.7, we present our analysis of the posting method, exploring the inherent assumptions in the system and identifying the problems it imposes on the schools and pupils. Finally, in section 4.8, recommendations for the MOE are provided.

4.4 Current Secondary School Education System in Singapore

On completion of their PSLE, pupils will be streamed to one of the following secondary courses based on their PSLE results: 1.) Special Course (S-Course); 2.) Express Course (E-Course); 3.) Normal (Academic) Course [N (A)-Course] and 4.) Normal (Technical) Course [N (T)-Course].

Both the S-Course and E-Course are four-year secondary courses. The S-Course is for pupils who are in the top 10% of the cohort. They will study English and Higher Chinese/Higher Malay/Higher Tamil. On the contrary, the E-Course pupils will study English and
Chinese/Malay/Tamil. However, they may study Higher Chinese/Higher Malay/Higher Tamil if they are in the top 11 - 20% of the PSLE pupils and have obtained an A* in the mother tongue language or a distinction in the higher mother tongue language, and at least an A in English. At the end of their courses, both the S-Course and the E-Course pupils will sit for the Singapore-Cambridge General Certificate of Education Ordinary (GCE `O') Level examination.

Compared to the S-Course and E-Course, the N (A)-Course is a four to five-year secondary school course. The N (A)-Course pupils will study two languages: English and Chinese/Malay/Tamil, plus a range of academic subjects. At the end of the fourth year, they will sit for the Singapore-Cambridge General Certificate of Education Normal (GCE `N') Level examination. Pupils who perform well in the examination will proceed to a fifth year of secondary education and sit for the GCE `O' Level examination at the end of the year. Those who are not eligible for the fifth year of education may take up technical-vocational education and training at the Institute of Technical Education (ITE).

Likewise, the N (T)-Course is a four to five-year secondary course. Pupils taking this course will study English, Basic Chinese/Basic Malay/Basic Tamil and a range of technical-oriented subjects. Like the N (A)-Course pupils, they will sit for the GCE `N' Level examination at the end of the fourth year. Those more able pupils may continue for an additional year and sit for the GCE `O' Level examination. Those who do not qualify to enter the fifth year may take up technical-vocational education and training at ITE.

4.5 Secondary One Posting Exercise

Presently, upon the completion of his Primary School Leaving Examination (PSLE), each primary six (P6) pupil in Singapore seeks his/her enrollment in secondary school through a
centralized system called the Secondary One Posting Exercise. The posting exercise was implemented by the MOE in 1984.

4.5.1 Prior to the posting

**First option form**

Before taking their PSLE in October, all P6 pupils, in around August, will be given an option form. In the option form, each pupil is to list down, in order of his preferences, the six secondary schools he/she would like to attend. The pupils or rather their parents are advised by the MOE to make realistic choices because they will not be allowed to make any change once they have indicated the choices in the option form. After the form is completed, it will be returned to the MOE.

**Second option form**

On the release of the PSLE results, some pupils will be given a second option. Pupils with outstanding performance in the PSLE will be invited to apply for the Edusave Entrance Scholarships for Independent Schools (EESIS). They will be given up to four Independent schools to choose from. These pupils are advised by the MOE to opt for more than one independent school, as vacancies in the independent schools are limited.

Likewise, the top 10% of the successful PSLE candidates who do Chinese and Higher Chinese will also be given a second option. In this option, they are given up to 3 Special Assistance Plan (SAP) schools to choose from.
For posting pupils to secondary schools, the choices of schools made in the second option are taken into account first before the choices of schools made in August.

### 4.5.2 Posting Process

The posting of pupils to secondary schools is (partially) computerized. As illustrated in Figure 4.1 which depicts the process of posting PSLE pupils to secondary schools, all pupils are ranked according to their aggregate PSLE scores regardless of their choices. Though some pupils may have the same aggregate scores, it is still possible to rank such pupils by referring to more detailed scores that can extend to several decimals places.

Subject to the vacancies in a school, a pupil with better performance will be considered for admission to that school first before the next pupil is considered. The order of posting each PSLE pupil to schools is as follows:

1.) Consider pupils who have applied for EESIS in the second option and post them to independent schools.

2.) Consider pupils who have applied for SAP schools in the second option and post them to SAP schools.

3.) Consider all other pupils one by one, and post them to schools chosen in August.

When a pupil fails to get into any of the schools of his choice, he/she will be posted manually to a neighborhood school that still has vacancies. If there is no such school available, the pupil will be posted manually to schools in other postal districts that still have vacancies.

As shown in Figure 4.1, in the process of posting the pupils to schools, a pupil who applies for his affiliated secondary school as his first choice will be given priority for entering
the school. However, not all pupils can gain admission to their affiliated secondary schools as the admission to the affiliated schools is subjected to the vacancies in these schools.

Each pupil will be informed of the school he/she is posted to by late December, just before the new school semester starts in January.

4.6 Formal Model

Our model is actually a reformulation of what is most often referred to in the theoretical literature as the “college admission” model (or rather what we call in our Academic Exercise as the hospitals/residents problem). The first elements of our model are two finite and disjoint sets, \( P = \{p_1, \ldots, p_m\} \) and \( S = \{s_1, \ldots, s_n\} \), of primary six pupils and secondary schools respectively. Each pupil has preferences over all the schools; likewise, each school has preferences over all the pupils.

The amount of positions each school can fill during the posting exercise is an integer \( q \) called quota. An outcome of the posting process is a matching of pupils to schools, such that each pupil is matched to at most one school, and each school is matched to at most \( q \) pupils. Note that the quota might differ for different school. A matching \( x \) is said to be unstable if there exists a pupil \( p_i \) and a school \( s_j \) such that 1.) \( p_i \) prefers \( s_j \) to his assigned school, and 2.) either \( s_j \) prefers \( p_i \) to one of its assigned pupils or \( s_j \) does not have all its vacancies filled in the matching.
4.7 Analysis

In this section, we present our analysis of the MOE current posting system. This analysis is undertaken in two parts. First and foremost, we examine the current system to explore the inherent assumptions. Finally, the current system is evaluated for the problems it imposes on the parents/pupils and the schools.

4.7.1 Assumptions of the Current Posting System

Based on the MOE posting system shown in Figure 4.1, we identified 4 implicit assumptions in the system.

1.) Observe that in the posting system, the preferences of schools are not considered. By ranking the pupils in order of merit, the system is assuming that the schools rank the pupils according to their performance in the PSLE. This assumption may not generalize to those affiliated secondary schools. As we can see in the second assumption, the preferences of these affiliated secondary schools are slightly different from those schools without affiliated primary schools.

2.) By giving priority to pupils who list their affiliated secondary school as the first choice, the system is assuming that these schools prefer these affiliated pupils to other pupils who may have a better PSLE results. This assumption is confirmed by a Strait Times article on 8/11/89 entitled “Independent schools will accept all who qualify”. As quoted by Brother Kevin Byrne, principal of St Joseph’s Institution (SJI), “Top priority is given to our feeder primary school pupils, and we take in those who are either in the special or the
express stream…. Anyone who makes it into either would get a place in the independent school if they opt for it”. (See Appendix A for the article).

3.) By manually posting pupils to neighborhood schools when they fail to get into any of their choices of schools, the system is assuming that pupils would prefer entering a school that is near their home to one that is in other postal districts. This is quite a reasonable assumption. According to a 1992 random survey by the Singapore Press Holdings’ research and information department among 313 parents and pupils, 30% of the respondents indicated that the school’s proximity to home is one very important factor influencing their decisions in choosing secondary schools. (See Appendix B for the article).

4.) By manually posting all the unplaced pupils to schools in other postal districts which still have vacancies, the system is assuming that these pupils are indifferent towards these schools.

4.7.2 Problems Identified in the Current Posting System

Reviewing the current MOE posting system, we found some significant problems, which we seek to overcome in our study. The current system fails to consider the emotion trauma the parents went through and the difficulties faced by them in formulating their options. It also fails to address the needs of the schools. These two aspects would be discussed in details as below.
4.7.2.1 Difficulties faced by the pupils/parents

The time around August every year is the most stressful period for the primary six pupils or more precisely, the pupils’ parents. This is because they need to decide which secondary school is best for their child. Choosing a right secondary school for their child is never an easy chore for the parents. Several factors such as the proximity of school from home, and the reputation of school are to be considered before a decision is made. Decision-making is further complicated by the fact that 1.) Parents are only given limited options to express their preferences and 2.) Parents have to make their decisions prior to the release of the PSLE results.

Parents in Singapore, in general, have high expectations for their children and would want them to get into the best secondary schools such as Raffles Institution or The Chinese High School. However, with only six choices given to them, and knowing that many good students will be applying to these schools, parents cannot afford to reveal their true preferences. These parents would realize that if they were to list down just the ideal top six schools as their six choices, they run the risk of not getting them at all if their child fails to qualify for these schools. Worse, this may actually result in their child getting into a school that is out of their consideration as manual posting would be carried when the child fails to get into any of the six choices. On the micro level, the child may end up spending four/five unhappy years in the school that he is forced to go to. On the macro level, the child’s future may be adversely affected.

The parents are thus forced to be more “realistic” when choosing the schools for their child. Ironically, in the interview with the ministry officials, it seems that the ministry expects the parents to do exactly that! Consequently, the main issue here is for the parents to optimally utilize the six options provided by the MOE. Notwithstanding, how are parents going to make
good use of their choices if they are uncertain how their child will fare relative to other candidates in the PSLE? [Strait Times, 17/4/96: See Appendix C for the article]).

Well, for one, the parents are expected to use their child’s past academic performance in the school as a gauge of his/her chances of entering a particular secondary school. However, it is important to realize that the ultimate determinant still rests on the child’s PSLE result. Therefore, asking the parents to choose a secondary school prior to the release of PSLE results is equivalent to asking them to speculate the schools their child can qualify for. This is definitely not going to be easy and it can cause tremendous agony for the parents.

The Ministry of Education tried to make life easier for the parents by providing them with information such as the schools’ entry cut off points for the previous year to help them make rational decisions. In recent years, they have also resorted to ranking all the schools in Singapore so that the parents will be better informed of the schools’ strengths and weaknesses. However, this does not address the parents’ major concern in formulating their choices: They have to submit their options way before the pupils take the examination!

We note that it is not possible for the Ministry to solicit the pupils’ options after the release of the examination results, due to the tight time span between the release of results and the start of the new school term.

4.7.3 Difficulties faced by the schools

In the posting exercise, the preferences of the schools are not considered. The schools have no say over the type of pupils they take in. They would just accept the pupils who have been assigned to them by the MOE.
In fact, the system actually implicitly assumes that all schools (with the exception of the affiliated schools) rank the pupils according to the PSLE results. This may not hold true after all. For example, some schools may actually prefer a pupil who is excellent in certain extra-curriculum activities (ECAs) say, badminton, which they are trying to promote. In addition, the PSLE result is certainly not equivalent to a student academic ability, as his/her performance in the examination can be affected by many other environmental factors.

In order to attract the desired pupils, many schools, especially those newly established and do not have much recognition among the public, have actually taken on advertising campaigns such as the organization of open houses, sending of brochures to parents and conducting talks at primary schools. Though such marketing efforts help, schools still depend significantly on the ST School 100 ranking to attract their desired pupils. According to a ST article on 23/4/94, entitled “Neighborhood schools’ good rankings draw better pupils”, the principal of Riverside Secondary School, Mr Oliver Balasingam actually confirmed the importance of ST ranking (see Appendix D for the article).

Although the ST 100 ranking is an important device for the schools to attract their desired pupils, over-reliance on it could induce the schools to over-emphasize the importance of academic results. This is clearly an undesirable repercussion as it actually impedes the Singapore government from achieving the “Thinking Schools, Leaning Nation” vision.

4.8 Recommendations

As supported by past studies [e.g. Roth (1984,1990)], it is critical and desirable that stability condition exists in the two-sided matching market. However, apparently, stability is not MOE ‘s major concern in the placement of P6 pupils in secondary schools. To ensure that
stability is always present in the market, we recommend a whole new posting system, which is based on the student-propose algorithm. This algorithm is actually a generalization of the current posting system. However, in order to implement this algorithm, the MOE needs to remove the restrictions on the number of choices given to each P6 pupil and give freedom to the schools to express their preferences for the pupils.

The benefits of our proposed system are as follows:

1.) By removing the restriction on the number of options each pupil has, this means that parents can list down 10 schools or 100 schools (if they desire!) according to their true preferences. They need not agonized over whether their child can qualify for the schools they have chosen. This is because if the parents are not confident of their child getting into their desired schools, they can always list down as many schools as they want. With the implementation of the student-propose algorithm, the MOE is assured that parents will reveal their true preferences as such an algorithm makes truth telling a dominant strategy for the parents.

2.) By allowing schools to express their preferences for the pupils, they are likely to get students of the caliber they desire. Better match of schools and students can then be expected. For instance, a school, which is trying to promote certain ECAs say athletics, could inform the ministry of their selection criterion and place more emphasis on athletics ability. In fact, currently, the Singapore government is emphasizing heavily on giving pupils an all-rounded education (i.e. ECA, creative studies and independent thinking are gaining importance). However, given the fact that in the Secondary One Posting Exercise, pupils are ranked by
merit according to their PSLE results, the current system actually runs counter to the government’s efforts in this area.

3.) Most importantly, the Ministry’s task would be greatly simplified since our proposed system virtually eliminates the need for the provision of the second option. In addition, it eliminates the need for manual posting method that slows down the current posting exercise. By de-emphasizing the importance of the PSLE results in the posting method, the Ministry can have more time to plan for the collection of information (i.e. pupils and schools preferences and requirements), rather than waiting until August each year. The current system is designed in such so that the pupils’ performance in their respective mid-term exam will be made known to the parents, who can then make more “realistic” expectations.

4.) There is an enormous amount of literature on the college admission problem that can be used to bear on the existing students-schools market. For instance, the student propose mechanism suggested here has recently been recommended to be the official posting method to assign interns to hospitals in the US [Roth (1997)]. The high participation rate in this interns/hospitals market, despite the fact that participation is voluntary, shows the relevance and importance of ensuring stability in the two-sided market. The current posting method used by the Ministry seems to place undue emphasis on the parents making realistic choices, which to many are agonizing due to the uncertainty in the students’ performance in the PSLE. The student-propose algorithm suggested in this thesis remove the needs for strategic manipulation by the parents, as the students are guaranteed to be assigned their best stable choices under the student propose algorithm. Also, as we have learnt in Chapter two, the
adoption of the student-propose algorithm would give incentives for some schools to falsify their true preferences. From Chapter three, we know that in the Gale-Shapley model (a one-to-one model with no rejection), the chances of a woman to benefit from falsifying her preferences are pretty slim when the man-propose algorithm is used. Although the primary-six pupils/secondary schools market is a many-to-one matching market, like the Gale-Shapley model, it is a market without rejection. Thus, we may expect the above result to hold for the market. Specifically, we may conclude that the chances of a school to benefit from falsifying its preferences are slim when the student-propose algorithm is used. This means that the MOE need not worry that the schools would falsify their preferences.

Despite the above advantages of our proposed system, there are however, some minor flaws in the proposed system that need to be overcome:

1.) By removing the restriction on number of options each pupil has, we expect an immense increase in the amount of data that MOE would need to handle. This is, however, a minor problem, which can be easily solved with the current state of technology. For instance, requesting that parents register their preferences over the web allows the information to be directly captured into the database.

2.) The proposed system is certainly much more complex than the existing one. Furthermore, the proposed system can be strategically manipulated by a coalition of student and school, although it is not likely to be manipulated by an individual. In view of this difficulty, the
schools should not be allowed to rank the pupils directly, but instead, reflect to the Ministry the desired criteria and a formula for the ranking of pupils.
5 CONCLUSION

5.1 Concluding Remarks

There are two main areas that we are exploring in our study. The first part of our study focuses on the theoretical aspects of the stable marriage problem. In specific, we seek to address the strategic issues in Gale-Shapley model (i.e. a model with no rejection, where each agent’s preference list must be complete) that has not been addressed prior to this thesis. We proposed a cheating heuristic for the women. Analogous to Gale and Sotomayor’s (1985b) cheating strategy that can only be used in the rejection model, our cheating heuristic is optimal in the sense that it gives each woman the best possible stable partner she can achieve when cheating can only be done via permuting her preferences and given that the man-propose algorithm is used. Interestingly, from our study, we discover that the strategic behavior of the women can be very different under the models with and without rejection.

Unlike in the rejection model where declaration of partners is allowed in the cheating strategy, it is exciting to note that a woman cannot always be assured of recovering her woman-optimal stable partner from the man-propose algorithm by cheating in the Gale-Shapley marriage game. In fact, implementation of our heuristic shows that the chances that a woman can benefit from cheating are extremely small in the Gale-Shapley game. Surprisingly, even if the women collude, their influence on the outcome of the game is still limiting. One very good example to illustrate this result is one where each woman is ranked first by exactly one man. In this case, there is no conflict among the men. The resulting man-optimal stable matching will always give each man his first choice regardless how the women rank the men in their list. Hence, by just requiring the women to submit a complete preference list, we are clearly restricting their
strategic options. This is in a sense good because it reduces the possibility that the women will cheat.

The second part of our study emphasizes on the importance of stability. As evidenced from past studies [Roth (1984,1990)], it is critical and desirable that stability exists in every two-sided matching market. The importance of stability is best highlighted in a voluntary system where acceptance of proposed matching is voluntary. Such system will collapse if stability is not in place. On the other hand, in a highly regulated or controlled market, the impact of instability is not as obvious.

To apply the knowledge of stable matching to a real two-sided matching market, we conducted a study of the Secondary One Posting Exercise in Singapore. This posting exercise is regulated by the MOE. As such, we are actually looking at a non-voluntary system.

From our analysis of the posting exercise/system, we deduce a lack of emphasis on stable matching on the part of the Ministry of Education (MOE). The MOE seems to be more interested in matching as many pupils to schools as possible, based on the pupils’ academic results. It does not seem to be concerned about the satisfaction of parents (in terms of getting the desired schools) or the specific needs of schools (in terms of the type of pupils they desire). In short, the MOE basically ignores the ‘emotional’ aspect of the pupil-school match. Despite the obvious disregard for stability (observed through the behavior of MOE), we would expect the current posting system to collapse if the pupils are allowed to request for transfer to another school after the posting result is out. MOE prevented the breakdown by simply stipulating that transfer is only allowed under special circumstances, and that the schools have to take in whatever pupils allocated to them.
Although one may argue that the MOE actually does an excellent job in maintaining stability (in matching schools and students) through its control, we feel that the main side-effect of this method of enforcing ‘stability’ is the resulting shift in the responsibility of achieving optimal match to the parents. Specifically, parents have to be more careful in making realistic choices of schools for their child under the current system. This would inevitably create tremendous stress and agony for the parents, as mentioned in detail in Chapter four. Besides this, schools are also disadvantaged in the sense that they are not able to get the pupils of their desire. On the whole, we feel that the lack of emphasis on stable matching is a big deficiency of the current system.

To rectify this deficiency, the only way is to ensure that stability always exist in the system. As such, we proposed, in our study, a completely new system of stable matching based on the student-propose algorithm (refer to Chapter four). We hope that this new system could help in reducing the emotional trauma or frustration felt by parents in the posting exercise, as well as liberating schools in the area of choice of pupils.

5.2 Directions for Future Research

In the theoretical literature, there are still many problems unsolved in the study of two-sided matching markets. For instance, what kind of lying is advantageous for an individual or coalition when an egalitarian stable matching is to be obtained? Subsequent research in two-sided matching markets may seek to address such issue.

There are many interesting two-sided matching markets in Singapore that are worth examining. One excellent situation that is worth studying is the Primary One Registration Exercise. Currently, there are numerous complaints coming from the parents regarding the
system. Why does that happen? It would be worthwhile to conduct an in-depth study on the current system, seeking for an explanation for the current happenings in the market.