Portfolio Value-at-Risk Optimization for Asymmetric Distributed Asset Returns

27 Dec 2008

Joel Goh
NUS Business School

with Kian-Guan Lim, Melvyn Sim, and Weina Zhang
Outline

- Introduction to VaR
- Robust Optimization of VaR
- Partitioned Statistics
- Computational Experiment
Outline

- Introduction to VaR
- Robust Optimization of VaR
- Partitioned Statistics
- Computational Experiment
Introduction: Problem statement

- How to allocate resources between $N$ different assets to minimize risk?

- Key problem: Define an appropriate measure of risk
  - Meaningful to problem and tractable

- Disaster relief, project management, inventory control, warehouse control, financial portfolio allocation
  - Different notions of “resources” or “assets”
Portfolio Risk - History

- Classical: Harry Markowitz (1952)
  - “Risk” – measured by the variance
  - Optimization problem: Find “optimal” weights so that risk is minimized for any given level of target portfolio return
  - LCQP

- Recent developments
  - High returns do not qualify as “risky,” try to prevent only low returns
  - Quantile-based risk measures, e.g., Value-at-Risk and Conditional Value at Risk
  - Measure of extreme outcomes
Value-at-Risk (VaR)

- $\text{VaR}_{(1-\varepsilon)} = V$
  - $\varepsilon$ usually small (~0.01)
  - We are $100 \times (1-\varepsilon)$% percent certain that we will not lose more than $V$
  - Intuitively speaking, “$\varepsilon$-worst-case” scenario
VaR Minimization

- Portfolio selection problem that minimizes $\text{VaR}_{1-\epsilon}$

$$\text{VaR}_{1-\epsilon}^* = \min \gamma$$

s.t.  $\mathbb{P}(\gamma + \tilde{r}'x \geq 0) \geq 1 - \epsilon$

$x \in X$

- Main issues
  - Generally hard to optimize
  - Chance constraints
  - Distributions of returns are not known
Coherent Risk Measures

- Axioms of Coherent Risk \((Artzner \textit{et al.}, \ 1999)\)
  1. Monotonicity
  2. Translational Invariance
  3. Convexity
  4. Positive Homogeneity

- VaR is not a coherent measure of risk

- Conditional VaR \((Rockafellar \textit{and} Uryasev, \ 2000)\)
  - Coherent Risk Measure
  - Measures the conditional expected loss
CVaR Minimization

- Portfolio selection problem that minimizes $\text{CVaR}_{1-\epsilon}$

$$CV\alpha R^*_{1-\epsilon} = \min \gamma$$

$$s.t. \quad v + \frac{1}{\epsilon} \mathbb{E}(-\tilde{r}'x - v)^+ \leq \gamma$$

$$x \in X, v \in \mathbb{R}$$

$$\text{VaR}^*_{1-\epsilon} \leq CV\alpha R^*_{1-\epsilon}$$

- Main issues
  - Distributions of returns are not known
Outline

- Introduction to VaR
- Robust Optimization of VaR
- Partitioned Statistics
- Computational Experiment
Optimization on families of distributions

- Know descriptive statistics of the distribution
  - Support, Mean, Covariance etc

- Optimize the worst-case over all distributions having these statistics
  - Optimal solution is *robust* against realizations of different distributions
  - Robust Optimization

- A more difficult problem?
Worst-case VaR

- A tractable instance (*El Ghaoui et al. 2003*)
  - Known mean return and covariance matrix of underlying assets
    \[ F_1 = \{ P : E_P [\tilde{r}] = \mu, E_P [(\tilde{r} - \mu)(\tilde{r} - \mu)^\prime] = \Sigma \} \]
    \[ WVaR_{1-\epsilon}^* = \min_{x \in X} -\mu'x + \sqrt{\frac{1 - \epsilon}{\epsilon}} \sqrt{x'\Sigma x} \]
  - For fixed target mean, reduces to Markowitz model

- If we have more information, can we improve on this?
  - What kind of information?
WVaR Analysis

- **WVaR**
  - Tractable (SOCP), BUT does not capture asymmetry
  - Because Mean, Covariance does not measure this

- **How to capture asymmetry?**
  - Third and fourth moments (skewness and kurtosis)
  - Problems:
    - Difficult to measure empirically, tends to be unstable
    - Harder to optimize
Outline

- Introduction to VaR
- Robust Optimization of VaR
- Partitioned Statistics
- Computational Experiment
Partitioned Statistics

- Consider partitioning returns into its positive and negative parts
  - Intuition: Isolate statistical information whenever returns are positives (gains) and negatives (losses) respectively.

\[
\tilde{r}_i^1 = \max\{\tilde{r}_i, 0\} = \tilde{r}_i^+ \\
\tilde{r}_i^2 = \min\{\tilde{r}_i, 0\} = -\tilde{r}_i^-
\]

\[
\mu^1 = \mathbb{E}(\tilde{r}^1) \\
\mu^2 = \mathbb{E}(\tilde{r}^2)
\]

\[
\hat{\Sigma} = \text{cov}(\tilde{r}^1, \tilde{r}^2)
\]
Partitioned Statistics Example

Example: 2 independent standard normal random variables

\[ \mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \hspace{1cm} \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ \mu^1 = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \hspace{1cm} \mu^2 = \begin{bmatrix} -0.4 \\ -0.4 \end{bmatrix} \]

\[ \hat{\Sigma} = \begin{bmatrix} 0.34 & 0 & 0.16 & 0 \\ 0 & 0.34 & 0 & 0.16 \\ 0.16 & 0 & 0.34 & 0 \\ 0 & 0.16 & 0 & 0.34 \end{bmatrix} \]
Partitioned VaR (PVaR)

- Partitioned statistics provide more information about distribution than regular mean / covariance
  - We can recover original portfolio statistics

\[ \mathbb{E}(\tilde{r}'x) = \mu'x = \mu_1'x + \mu_2'x \]

\[ \text{var}(\tilde{r}'x) = x'\Sigma x = (x'x')\hat{\Sigma} \left( \begin{array}{c} \mu_x \\ \mu_x \end{array} \right) = \left\| \begin{array}{c} \mu_x \\ \mu_x \end{array} \right\|_{\hat{\Sigma}}^2 \]

- How to make use of this information?
Partitioned VaR (PVaR)

- Define

\[
PV aR^*_1 - \epsilon = \min \left\{ -\mu'x + \sqrt{\frac{1 - \epsilon}{\epsilon}} \left\| \Sigma (x-r) \right\|_2 + \mu^1' r - \mu^2' s \right\}
\text{ s.t. } \begin{align*}
r &\geq 0, \ s \geq 0 \\
x &\in X
\end{align*}
\]

- Can be shown that

\[
VaR^*_1 - \epsilon \leq PV aR^*_1 - \epsilon \leq WV aR^*_1 - \epsilon
\]

- If partitioned statistics are known, we can get a more accurate estimate of the VaR than the WVaR
- Can be made coherent if we have support information
Outline

- Introduction to VaR
- Robust Optimization of VaR
- Partitioned Statistics
- Computational Experiment
Simulated Experiment 1

- 10 independent assets with equal mean, equal variance

- Skewnormal distribution with increasing negative skew
  - More negative skew: fatter tail, poorer tail-end performance

- Optimized under 2 different objectives
  - MAX Sharpe Ratio (mean / Standard Deviation)
  - MAX Normalized PVaR (mean / PVaR)

- Optimized using ROAM:
  - In-house modeling software for Robust Optimization
Simulated Experiment 1

Portfolio allocation weights for 10 increasingly skewed assets
under different optimization objectives

- Sharpe Ratio
- 99.0% PVaR
Simulated Experiment 2

- 5 independent assets with same std deviation (0.03)
- Increasing mean, increasing negative skew

\[
\begin{align*}
\mu_i &= 0.01 + 0.0025i \\
\delta_i &= -0.25i \\
\gamma_i &= \frac{4-\pi}{2} \frac{\delta_i^3}{(\frac{\pi}{2}-\delta_i^2)^{3/2}} \quad \forall i \in \{0, 1, 2, 3, 4\}
\end{align*}
\]

- For different target returns, separately optimize under the Markowitz and PVaR objectives
  - Investigate Efficient Frontier
Simulated Experiment 2

Out-of-sample Return vs Std Deviation

Out-of-sample Return vs 99.0% Partitioned Value-at-Risk

Out-of-sample Return vs 99.0% Empirical Value-at-Risk

Portfolio Weights

Weight in Portfolio vs Asset Number
Empirical Experiment

- 48 Industry returns from July 1, 1963 to August 31, 2007

- Rolling-window procedure
  - 1 period = ½ year
  - Use data from past 10 periods to compute partitioned statistics
  - Optimize portfolio and hold for 1 period, store out-of-sample realized returns
  - Move forward by 1 period and repeat

- Similar to previous, vary target return and observe the efficient frontier
Empirical Experiment

Out-of-sample Return vs 99.0% Empirical Value-at-Risk

- P VaR Optimized
- Mean-Variance Optimized
Conclusion

- Partitioned Statistics
  - To capture asymmetry in distribution of primitive uncertainties
  - Instead of higher order moments

- Introduce concept of PVaR
  - Tighter bound on VaR than WVaR

- Computational Experiments
  - Simulated and Real Data
  - Promising results that our method can improve over classical methods